FREE TRANSVERSE VIBRATIONS
OF RECTANGULAR LAMINATED PLATES

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Chien-Chang Lin

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Approved:

Wilton W. King, Chairman

Dar-Veig Ho

David J. McGill

James T. S. Wang

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LIST OF SYMBOLS

a, b, h	Plate dimensions, in
u, v, w	Components of displacement, in
ρ_o	Mass density, lb-sec^2/in^4
$\sigma_{xx}, \sigma_{yy}, \tau_{xy}$	Engineering stress components, lb/in^2
$\epsilon_{xx}, \epsilon_{yy}, \gamma_{xy}$	Engineering strain components
K_{xx}, K_{yy}, K_{xy}	Curvature of middle surface, 1/in
E_ℓ, E_t	Moduli of a unidirectional composite parallel and perpendicular to the fiber direction, lb/in^2
$G_{\ell t}$	Longitudinal shear modulus of a unidirectional composite in x-y plane, lb/in^2
$\nu_{\ell t}, \nu_{t\ell}$	Poisson's ratios of unidirectional composite
$[C^*_{ij}]$	Reduced stiffness matrix of a constituent layer, lb/in^2
$N_{xx}, N_{yy}, N_{xy}, Q_x, Q_y$	Stress resultants, lb/in
M_{xx}, M_{yy}, M_{xy}	Stress couples, lb - in/in
$\bar{U}_s$	Strain energy density, lb/in^2
t	Time, sec
ω, Ω_i	Circular frequencies, cycles/sec
θ	Angle of the fiber direction with respect to the x-axis of the plate, rad
D_x, D_y, H	Rigidity constants in a homogeneous orthotropic plate, lb/in^2
ν_{xy}, ν_{yx},	Poisson's ratios in a homogeneous orthotropic plate
ν	Poisson's ratio in a homogeneous isotropic plate
$L_{ij}, \ldots.$	Linear differential operators

$\{\varphi\}$	Column matrix of 3 x 1
k_1, k_2	Wave numbers in the x- and y-directions, 1/in
ℓ_x, ℓ_y	Half-wavelengths in the x- and y-directions, in
s_x, s_y	Numbers of nodal lines running "perpendicular" to the x- and y-directions

SUMMARY

This dissertation is an analytical study of vibrations of laminated rectangular thin plates.

By using the Kirchhoff assumptions in linear plate theory, governing equations of laminated plates developed by Whitney and Leissa are verified by a derivation from energy principles. The types of laminated plates under investigation are unsymmetric cross-ply and antisymmetric angle-ply rectangular plates for which solutions by separation of variables are possible.

The inplane inertia effects on natural frequencies of both types of laminated plates are investigated for the case of all simply-supported edges. These effects are not significant for the predominantly flexural mode which is of interest in the present study.

Closed form solutions for natural modes, frequencies, and modal stresses are obtained for both types of laminated plates neglecting inplane inertias with two opposite edges simply-supported. The method provides a fast technique of computation for the solutions of eigenvalue problems.

An asymptotic method based on the concept of dynamic edge effects is applied to determine natural modes, frequencies, and modal stresses of rectangular plates. The utility of this method is discussed in detail. In contrast to other approximate methods, the asymptotic method gives more accurate solutions for higher modes and greatly reduces the computational effort.

The effects of bending-stretching coupling on laminate response are investigated for a wide variety of boundary conditions. The effects of inplane boundary conditions on laminate response and the applicability of the cylindrical bending solution to modal vibrations are also explored. Some interesting phenomena are found.

CHAPTER I

INTRODUCTION

The need for high strength, high stiffness, and low density materials for use in modern aircraft and space vehicle structures has stimulated interest in fiber-reinforced composite materials.

A basic lamina of a fibrous composite can be considered as orthotropic and homogeneous with two principal material directions parallel and perpendicular to the direction of the fibers. When two or more orthotropic laminas are bonded together to form a composite laminate, this laminate is generally transversely heterogeneous.

The general theory of unsymmetric laminated rectangular plates within the classical Kirchhoff assumptions has been discussed by a number of investigators. In 1953, Smith [1] discussed the bending behavior of a two-layer laminated plate, in which the axes of elastic symmetry of two adjacent layers make angles $+\theta$ and $-\theta$ with the plate axes. He concluded that the laminate will behave as a homogeneous orthotropic plate. In 1961, the work by Reissner and Stavsky [2] showed this conclusion to be incorrect. Later investigations by Dong, Pister and Taylor [3], Azzi and Tsai [4], and Whitney and Leissa [5] substantiated the conclusion of [2] that there exists coupling between transverse bending and inplane stretching if laminates are layered up unsymmetrically about the middle plane.

Azzi and Tsai have demonstrated the applicability of classical plate theory to the prediction of elastic moduli of laminated composites. Reissner and Stavsky formulated the laminated plate problem in terms of a stress function and the transverse deflection governed by two coupled differential equations, and obtained solutions for an infinite two-layer plate subjected to different types of transverse and inplane static loadings. Dong, Pister and Tayer used the same formulation for a more general class of laminated plates. Incorporating the von Karman theory of large deflection of plates in addition to the classical Kirchhoff assumptions, Whitney and Leissa developed the general governing equations of laminated rectangular plates in terms of displacements including inertia terms and thermal stresses. In their analysis, closed form solutions to the linearized differential equations, excluding external shear tractions, thermal effects as well as inplane and rotary inertias, were obtained for static deflection, vibration and buckling of antisymmetric cross-ply and angle-ply plates with all edges simply-supported. However, solutions to other boundary value problems of thin laminated plates within classical Kirchhoff assumptions of small deflections are still very limited.

Applying the Ritz energy method and beam characteristic functions, Bert and Mayberry [6] analyzed the free vibrations of laminated rectangular plates with all rigidly-clamped edges. Solutions for four types of laminated plates were obtained for comparison with their experimental results. Using the Fourier series method, Whitney [7,8] investigated the effect of bending-extensional coupling and the effect of boundary conditions on static deflection, vibration and buckling of antisymmetric cross-ply and angle-ply rectangular plates. Pryor and Barker [9] presented some numerical results for the static deflections of antisymmetric angle-ply plates with all simply-supported edges and with all rigidly-clamped edges by the application of the finite element technique.

Employing a formulation analogues to the Levy's solutions for isotropic plates, Kan and Ito [10] obtained solutions for the static deflections of antisymmetric cross-ply plates with simple-supports at one pair of opposite edges and certain cases of boundary conditions at the other two edges.

Ashton [11] developed a "reduced bending stiffness method" for the analysis of unsymmetric laminates. In this method, the bending stiffnesses of a homogeneous anisotropic plate are replaced by the "reduced bending stiffnesses" of an unsymmetric laminate so that the problems of unsymmetric laminates can be solved in the same formulations as homogeneous plates. It has been shown [7,8] that the "reduced bending stiffness method" does not give acceptable agreement with coupled laminate solutions for certain orientations of antisymmetric angle-ply plates.

The present study is concerned with the theoretical investigation of free vibrations of laminated rectangular thin plates with small deformation. Using Kirchhoff assumptions in conjunction with the small deflection theory, governing equations [5] of laminated plates are derived directly from energy principles. The eigenvalue problems of laminated plates to be investigated are such that the twist-coupling terms are not involved so that solutions by separation of variables are possible [12]. These problems are divided into three classes as follows: (1) an analysis of inplane inertia effects for simply-supported laminated plates, (2) closed form solutions for laminated plates with two opposite sides simply-supported neglecting inplane inertias, (3) the analysis of laminated-plate vibrations using an asymptotic method including a special formulation for uncoupled laminates (also for homogeneous orthotropic plates). The effect of bending-stretching coupling on the response of laminated rectangular plates is ascertained for a wide variety of material parameters. The effect of inplane boundary conditions on the response of laminated rectangular plate is also treated.

The major emphasis of this study is on the application of the asymptotic method to determine the natural modes, frequencies, and modal stresses of laminated rectangular plates. This asymptotic method was previously developed and applied by Bolotin [13,14,15,16 and 17] to the vibrations of homogeneous isotropic plates, shells and rectangular parallelepipeds. Application to the homogeneous orthotropic clamped-plate was also briefly discussed [14]. According to this method, frequencies are determined as functions of the wave numbers and natural modes are expressed as a sum of the generating solution and the corrective solution which has been called the "dynamic edge effect." Computation using the Bolotin method is much simpler and more rapid compared with other approximate methods. However, this method is applicable for eigenvalue problems only if separation of variables is possible and the dynamic edge effect does not degenerate.

CHAPTER II

FORMULATION OF PROBLEMS FOR THE LAMINATED RECTANGULAR THIN PLATES

Basic Assumptions

A rectangular Cartesian coordinate system (x, y, z), as shown in Figure 1, such that the x- and y-axes lie in the middle plane and parallel to edges of the undeformed rectangular plate is used in deriving the equations. The displacement of an arbitrary point (x, y, z) in the plate is denoted by (u, v, w) and the displacement of a point (x, y, 0) at the middle plane is denoted by (u^o, v^o, w^o).

In many practical applications of plates the tractions acting on surfaces parallel to the middle surface are small compared with the bending or stretching stresses. When a plate is very thin, the smallness of tractions on the external surfaces suggests the smallness of tractions on any surface parallel to the middle surface. Thus the stress components σ_{zz}, τ_{yz}, τ_{xz} are small throughout the plate, and an approximate state of plane stress exists.

Assumptions concerning the material properties and the behavior of the laminated plates to be investigated are made as the following:

1. The plate is constructed of n layers of homogeneous orthotropic laminas bonded together. However, the directions of elastic symmetry —— directions parallel and perpendicular to the fibers —— of an individual layer need not be parallel to the x- and y-axes of the plate.

2. The material of each layer obeys Hooke's law.
3. The thickness is constant throughout the plate.
4. The plate is thin, i.e., the thickness h is much smaller than the least lateral dimension.
5. The plate deformation is small. Thus strain components ε_{xx}, ε_{yy}, γ_{xy} are small and displacements are small enough that linear strain-displacement equations may be used.
6. The normal stress σ_{zz} can be disregarded.
7. Every linear element normal to the middle surface of the undeformed plate does not change in length, and remains straight and normal to the deformed middle surface.

Assumption 7 is known as the "Kirchhoff hypothesis" in the theory of plates. According to this assumption the transverse shear strains γ_{yz} and γ_{xz} are zero. As a result, the transverse displacement w is independent of the z-coordinate and the inplane displacements, u, v are linear functions of z.

Strain-Displacement Relations

Following assumption 5, the strain-displacement relations can be written as follows:

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} u,_x \\ v,_y \\ w,_z \\ v,_z + w,_y \\ u,_z + w,_x \\ u,_y + v,_x \end{bmatrix} \tag{2-1}$$

where commas in the subscripts denote partial differentiation. According to assumption 7, the displacement functions are

$$\left\{u\right\} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} u^o \\ v^o \\ w^o \end{bmatrix} - z \begin{bmatrix} w,_x \\ w,_y \\ 0 \end{bmatrix} = \left\{u\right\}_o - z\left\{\varphi\right\} \tag{2-2}$$

where u^o, v^o and w^o are functions of x, y, and the time t. The brace $\{\ \}$ is used to indicate a column matrix (3x1 throughout this study).

Substituting (2-2) into (2-1) gives the result

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \varepsilon^o_{xx} \\ \varepsilon^o_{yy} \\ \gamma^o_{xy} \end{bmatrix} + z \begin{bmatrix} K_{xx} \\ K_{yy} \\ K_{xy} \end{bmatrix} \tag{2-3}$$

where $[K_{xx}\ K_{yy}\ K_{xy}] = -[w,_{xx}\ w,_{yy}\ 2w,_{xy}]$ (2-4)

Eq. (2-3) can be written in abbreviated form as

$$\left\{\varepsilon\right\} = \left\{\varepsilon^o\right\} + z\left\{K\right\} \tag{2-5}$$

Constitutive Equations

The generalized Hooke's law for a linearly elastic material is

$$\sigma_{ij} = C_{ijk\ell}\, e_{k\ell} \tag{2-6}$$

where σ_{ij} is the stress tensor, $e_{k\ell}$ is the strain tensor, and $C_{ijk\ell}$ is the tensor of the elastic constants of the material. Assuming a state of plane stress and an orthotropic material, then (denoting $\sigma_{xx} = \sigma_{11}$, $\sigma_{yy} = \sigma_{11}$, $\tau_{xy} = \sigma_{12}$, $\varepsilon_{xx} = e_{11}$, $\varepsilon_{yy} = e_{22}$, $\gamma_{xy} = 2e_{12}$) the stress-strain relations at any point in a lamina of the plate can be written in the form

$$\{\sigma\} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} C^*_{11} & C^*_{12} & C^*_{16} \\ C^*_{12} & C^*_{22} & C^*_{26} \\ C^*_{16} & C^*_{26} & C^*_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = [C^*]\{\varepsilon\} \tag{2-7}$$

The six components of stiffness matrix $[C^*]$ can be determined once the four elastic constants of orthotropic material are known [18], (Appendix A).

Stress resultants and stress couples of plates are defined as follows:

$$[N_{xx}\ N_{yy}\ N_{xy}] = \int_{-h/2}^{h/2} [\sigma_{xx}\ \sigma_{yy}\ \tau_{xy}]\, dz \tag{2-8}$$

$$[Q_x\ Q_y] = \int_{-h/2}^{h/2} [\tau_{xz}\ \tau_{yz}]\, dz \tag{2-9}$$

$$[M_{xx}\ M_{yy}\ M_{xy}] = \int_{-h/2}^{h/2} [\sigma_{xx}\ \sigma_{yy}\ \tau_{xy}]\, zdz \tag{2-10}$$

These stress resultants and stress couples are illustrated in Figure 1. Substituting (2-7) into (2-9) and (2-10) and incorporating Eq. (2.3) gives

$$\begin{bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \\ M_{xx} \\ M_{yy} \\ M_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon^{o}_{xx} \\ \varepsilon^{o}_{yy} \\ \gamma^{o}_{xy} \\ K_{xx} \\ K_{yy} \\ K_{xy} \end{bmatrix} \tag{2-11}$$

where $(A_{ij}, B_{ij}, D_{ij}) = \int_{-h/2}^{h/2} C^{*}_{ij} \, (1, z, z^2) \, dz$ (2-12)

The constitutive equation (2-11) can be written in compact matrix form

$$\begin{bmatrix} \{N\} \\ \{M\} \end{bmatrix} = \begin{bmatrix} [A] & [B] \\ [B] & [D] \end{bmatrix} \begin{bmatrix} \{\varepsilon^{o}\} \\ \{K\} \end{bmatrix} \tag{2-13}$$

Stiffness matrices [A], [B], and [D] are all symmetric. In the general case there are 18 plate-stiffness constants and, obviously, there exists a coupling between bending and stretching in the reference plane. If [B] is a zero matrix, e.g. C^{*}_{ij} (i, j = 1, 2, 6) are even functions of z, the coupling vanishes.

Equations of Motion

The general equations of motion in terms of displacements of reference plane for a laminated rectangular thin plate have been formulated by Whitney and Leissa [5] applying Newton's law of motion to the layered composite material. In the present analysis, these same equations of motion for a laminated rectangular plate within classical Kirchhoff assumptions will be derived from the energy point of view.

For small deformations, the model of an elastic body is taken to be a continuous body which obeys the generalized Hooke's law (2-6) and exists a unique unstressed (or unstrained) state. For this linear elastic body under isothermal or adiabatic conditions, there exists a strain energy density function $\bar{U}_s$ such that [19]

$$\bar{U}_s = \frac{1}{2} C_{ijk\ell}\, e_{ij}\, e_{k\ell} \tag{2-14}$$

with the property

$$\frac{\partial \bar{U}_s}{\partial e_{ij}} = \sigma_{ij} \tag{2-15}$$

Thus, the variation of the strain energy density is

$$\delta \bar{U}_s = \sigma_{ij}\, \delta e_{ij} \tag{2-16}$$

Assuming a state of plane stress and employing assumption 7, Eq. (2-16) can be written in the form

$$\delta \bar{U}_s = \{\sigma\}^T \delta \{\varepsilon\} \tag{2-17}$$

where the superscript T denotes the transpose of a matrix. Using Eqs. (2-5), (2-8) and (2-10), the variation of the total strain energy of a laminated rectangular thin plate is

$$\delta U_s = \int_V \delta \bar{U}_s dV = \int_0^b \int_0^a (\{N\}^T \delta \{\varepsilon^o\} + \{M\}^T \delta \{K\})\, dxdy \tag{2-18}$$

The kinetic energy of the laminated plate during free vibration is

$$T = \frac{1}{2}\int_o^b\int_o^a\int_{-h/2}^{h/2} \rho_o \{\dot{u}\}^T\{\dot{u}\} \quad dxdydz$$

$$= \int_o^b\int_o^a\left[\frac{1}{2}\rho\{\dot{u}\}_o^T\{\dot{u}\}_o - Q\{\dot{u}\}_o^T\{\dot{\phi}\} + \frac{1}{2}I\{\dot{\phi}\}^T\{\dot{\phi}\}\right]dxdy \quad (2\text{-}19)$$

where a dot on the top denotes time derivative, ρ_o is the mass density, and

$$(\rho,\ Q,\ I) = \int_{-h/2}^{h/2} \rho_o(1,\ z,\ z^2)\ dz \quad (2\text{-}20)$$

Using Hamilton's principle and the variational method, the equations of motion and boundary conditions for free vibrations of laminated rectangular thin plates can be obtained (Appendix B). The governing equations in terms of displacements are given by

$$[L]\{u\}_o = [\bar{M}]\{\ddot{u}\}_o \quad (2\text{-}21)$$

where

$$[L] = [L^*] + [L^{**}] + [L^+] + L^{++}] \quad (2\text{-}22)$$

$[L^*]$ and $[L^{**}]$ are 3 x 3 symmetric matrices; $[L^+]$, $[L^{++}]$ and $[\bar{M}]$ are 3 x 3 skew-symmetric matrices; also define

$$L^*_{11} = A_{11}(\ )_{,xx} + A_{66}(\ )_{,yy} \quad (2\text{-}23)$$

$$L^*_{12} = (A_{12} + A_{66})(\ )_{,xy}$$

$$L^*_{22} = A_{66}(\ )_{,xx} + A_{22}(\ )_{,yy}$$

$$-L^*_{33} = D_{11}(\ )_{,xxxx} + 2(D_{12} + 2D_{66})(\ )_{,xxyy} + D_{22}(\ )_{,yyyy}$$

$$L^{**}_{11} = 2A_{16}(\ )_{,xy}$$

$$L_{12}^{**} = A_{16}(\ \ ),_{xx} + A_{26}(\ \ ),_{yy}$$

$$L_{22}^{**} = 2A_{26}(\ \ ),_{xy}$$

$$-L_{33}^{**} = 4D_{16}(\ \ ),_{xxxy} + 4D_{26}(\ \ ),_{xyyy}$$

$$-L_{13}^{+} = B_{11}(\ \ ),_{xxx} + (B_{12} + 2B_{66})\ (\ \ ),_{xyy}$$

$$-L_{23}^{+} = (B_{12} + 2B_{66})\ (\ \ ),_{xyy} + B_{22}(\ \ ),_{yyy}$$

$$-L_{13}^{++} = 3B_{16}(\ \ ),_{xxy} + B_{26}(\ \ ),_{yyy}$$

$$-L_{23}^{++} = B_{16}(\ \ ),_{xxx} + 3B_{26}(\ \ ),_{xyy}$$

$$\bar{M}_{11} = \bar{M}_{22} = \rho$$

$$\bar{M}_{33} = \rho - I(\ \ ),_{xx} - I(\ \ ),_{yy}$$

$$-\bar{M}_{13} = Q(\ \ ),_{x}$$

$$-\bar{M}_{23} = Q(\ \ ),_{y}$$

and all other elements of differential operators are zero.

In order to investigate the coupling phenomenon between bending and stretching in a fiber-reinforced composite, attention will be focused on laminate arrangements for which transverse deflections are necessarily accompanied by stretching in the middle plane even in the classical linear theory of plates. Two types of layered structures are considered; they are cross-ply and antisymmetric angle-ply rectangular plates. For a cross-ply laminate the thickness, material properties and direction of fibers paralleling the plate edges (as shown in Figure 3) of each layer can be different. Antisymmetric angle-ply laminate is composite of pairs of laminas such that the two laminas of any particular pair having the same thickness and material properties are arranged symmetrically about the middle plane with orientation angles at $+\theta$ and $-\theta$, respectively, as shown in Figure 2.

For both types of plates, it can be shown that

$$A_{16} = A_{26} = D_{16} = D_{26} = 0 \tag{2-24}$$

Furthermore, in the case of unsymmetric cross-ply laminate it is found that $B_{16} = B_{26} = 0$. Thus, the governing equations of free vibrations for cross-ply thin plates neglecting rotary inertias are given by

$$[L^c]\{u\}_o = \rho[\delta_{ij}]\{\ddot{u}\}_o \tag{2-25}$$

where

$$[L^c] = [L^*] + [L^+] \tag{2-26}$$

and δ_{ij} is the Kronecker delta.

For antisymmetric angle-ply plates, all elements of the [B] matrix vanish except B_{16} and B_{26}, and the governing equations of free vibrations for antisymmetric angle-ply thin plates neglecting rotary inertias are given by

$$[L^a]\{u\}_o = \rho[\delta_{ij}]\{\ddot{u}\}_o \tag{2-27}$$

where

$$[L^a] = [L^*] + [L^{++}] \tag{2-28}$$

Boundary Conditions

Obviously, the equations of free vibration for an unsymmetric laminated rectangular thin plate are coupled between the inplane displacements and the transverse deflection of reference (middle) plane.

To obtain solutions to boundary value or eigenvalue problems, four boundary conditions at each edge are needed. The boundary conditions for both cross-ply and anti-symmetric angle-ply rectangular thin plates are prescribed as follows:

(i) along $x = 0$ and $x = a$

either (geometric) or (natural)

$u^o = 0$ $$N_{xx} = A_{11}u^o_{,x} + A_{12}v^o_{,y} - B_{11}w_{,xx} - 2B_{16}w_{,xy} - B_{12}w_{,yy} = 0 \tag{2-29}$$

$v^o = 0$ $$N_{xy} = A_{66}(u^o_{,y} + v^o_{,x}) - B_{16}w_{,xx} - B_{26}w_{,yy} - 2B_{66}w_{,xy} = 0$$

$w = 0$ $$V_x = M_{xx,x} + 2M_{xy,y} = B_{11}u^o_{,xx} + 3B_{16}u^o_{,xy} + 2B_{66}u^o_{,yy} + B_{16}v^o_{,xx} + (B_{12} + 2B_{66})v^o_{,xy} + 2B_{26}v^o_{,yy} - D_{11}w_{,xxx} - (D_{12} + 4D_{66})w_{,xyy} = 0$$

$w_{,x} = 0$ $$M_{xx} = B_{11}u^o_{,x} + B_{12}v^o_{,y} + B_{16}(u^o_{,y} + v^o_{,x}) - D_{11}w_{,xx} - D_{12}w_{,yy} = 0$$

(ii) along $y = 0$ and $y = b$

either (geometric) or (natural)

$v^o = 0$ $$N_{yy} = A_{12}u^o_{,x} + A_{22}v^o_{,y} - B_{12}w_{,xx} - B_{22}w_{,yy} - 2B_{26}w_{,xy} = 0 \tag{2-30}$$

$$u^o = 0 \qquad N_{xy} = A_{66}(u^o_{,y} + v^o_{,x}) - B_{16}w_{,xx} - 2B_{26}w_{,yy}$$
$$- 2B_{66}w_{,xy} = 0$$

$$w = 0 \qquad V_y = M_{yy,y} + 2M_{xy,x} = 2B_{16}u^o_{,xx} + (B_{12}$$
$$+ 2B_{66})u^o_{,xy} + B_{26}u^o_{,yy} + 2B_{66}v^o_{,xx}$$
$$+ 3B_{26}v^o_{,xy} + B_{22}v^o_{,yy} - (D_{12} + 4D_{66})w_{,xxy}$$
$$- D_{22}w_{,yyy} = 0$$

$$w_{,y} = 0 \qquad M_{yy} = B_{12}u^o_{,x} + B_{22}v^o_{,y} + B_{26}(u^o_{,y} + v^o_{,x})$$
$$- D_{12}w_{,xx} - D_{22}w_{,yy} = 0$$

It is noted that $B_{16} = B_{26} = 0$ for cross-ply plates and $B_{11} = B_{22} = B_{12} = B_{66} = 0$ for antisymmetric angle-ply plates.

At any edge, say $x = 0$, there are four possibilities of inplane boundary conditions described as follows:

$$\begin{aligned} &(i) \quad u^o = v^o = 0 \\ &(ii) \quad N_{xx} = v^o = 0 \\ &(iii) \quad u^o = N_{xy} = 0 \\ &(iv) \quad N_{xx} = N_{xy} = 0 \end{aligned} \tag{2-31}$$

According to the types of edge constraints the boundary conditions at any edge, say $x = 0$, can be classified as the following three categories.

(1) Simple support: if $w = M_{xx} = 0$. There are four kinds of simple support which are to be designated by S1, S2, S3, and S4 corresponding to the inplane boundary conditions (i), (ii), (iii),

and (iv).

(2) Clamped support: if $w = w,_x = 0$. There are also four kinds of clamped support which are to be designated by C1, C2, C3, and C4 corresponding to the inplane boundary conditions (i), (ii), (iii), and (iv). The edge is said rigidly-clamped if the boundary conditions is the kind C1.

(3) Free: If $N_x = N_{xy} = V_x = M_{xx} = 0$, which is to be denoted by F.

CHAPTER III

INPLANE INERTIA EFFECTS FOR UNSYMMETRIC LAMINATED RECTANGULAR PLATES

Analytical Solutions

Closed form solutions for static deflection, vibration, and buckling of antisymmetric cross-ply and angle-ply rectangular thin plates with all edges simply-supported were obtained by Whitney and Leissa [5], in which the boundary conditions considered are specified as follows:

(1) For cross-ply plate

$$\text{at} \quad x = 0 \quad \text{and} \quad x = a : N_{xx} = v^{o} = w = M_{xx} = 0 \qquad (3\text{-}1)$$

$$\text{at} \quad y = 0 \quad \text{and} \quad y = b : u^{o} = N_{yy} = w = M_{yy} = 0 \qquad (3\text{-}2)$$

(2) For angle-ply plate

$$\text{at} \quad x = 0 \quad \text{and} \quad x = a : u^{o} = N_{xy} = w = M_{xx} = 0 \qquad (3\text{-}3)$$

$$\text{at} \quad y = 0 \quad \text{and} \quad y = b : N_{xy} = v^{o} = w = M_{yy} = 0 \qquad (3\text{-}4)$$

Inplane inertias in their solutions were not included.

In order to investigate the significance of inplane inertia effects on the natural frequencies of unsymmetric laminated rectangular plates, inplane-inertia terms are not considered for the cases of all simple supports satisfying boundary conditions (3-1) and (3-2) for cross-ply plates and (3-3) and (3-4) for antisymmetric angle-ply plates. For an unsymmetric cross-ply plate whose governing equations are given by (2-25), the displacement field for a free vibration mode can be assumed as follows:

$$\begin{bmatrix} u^o \\ v^o \\ w \end{bmatrix} = \begin{bmatrix} (A \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b}) \sin \omega t \\ (B \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b}) \sin \omega t \\ (C \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}) \sin \omega t \end{bmatrix} \tag{3-5}$$

It can be seen that all the boundary conditions (3-1) and (3-2) will be satisfied by the above displacement field. Putting Eq. (3-5) into the governing equation (2-25) gives the following set of equations in matrix form

$$\begin{bmatrix} Q_{11} - \lambda & Q_{12} & Q_{13} \\ Q_{12} & Q_{22} - \lambda & Q_{23} \\ Q_{13} & Q_{23} & Q_{33} - \lambda \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{3-6}$$

where

$$\begin{aligned} Q_{11} &= A_{11}k_1^2 + A_{66}k_2^2 \\ Q_{12} &= (A_{12} + A_{66})\, k_1 k_2 \\ Q_{22} &= A_{66}k_1^2 + A_{22}k_2^2 \\ Q_{13} &= -\, k_1\, [B_{11}k_1^2 + (B_{12} + 2B_{66})\, k_2^2] \\ Q_{23} &= -\, k_2\, [(B_{12} + 2B_{66})\, k_1^2 + B_{22}k_2^2] \\ Q_{33} &= D_{11}k_1^4 + 2(D_{12} + 2D_{66})k_1^2 k_2^2 + D_{22}k_2^4 \end{aligned} \tag{3-7}$$

where $K_1 = m\pi/a$, $K_2 = 2\pi/b$, and, $\lambda = \rho\,\Omega^2$. For a nontrivial solution, the determinant of the coefficient matrix in (3-6) must vanish to yield the following frequency equation

$$\lambda^3 - H_1\lambda^2 + H_2\lambda - H_3 = 0 \tag{3-8}$$

where

$$\begin{aligned} H_1 &= Q_{11} + Q_{22} + Q_{33} \\ H_2 &= Q_{11}Q_{22} + Q_{11}Q_{33} + Q_{22}Q_{33} - Q_{12}^2 - Q_{13}^2 - Q_{23}^2 \\ H_3 &= Q_{11}Q_{22}Q_{33} + 2Q_{12}Q_{13}Q_{23} - Q_{11}Q_{23}^2 - Q_{22}Q_{13}^2 - Q_{33}Q_{12}^2 \end{aligned} \tag{3-9}$$

Similarly for an antisymmetric angle-ply plate whose governing equations are given by (2-27) and boundary conditions are described as (3-3) and (3-4), the solution for the displacement field will be

$$\begin{bmatrix} u^o \\ v^o \\ w \end{bmatrix} = \begin{bmatrix} (E \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b}) \sin \Omega t \\ (F \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b}) \sin \Omega t \\ (G \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}) \sin \Omega t \end{bmatrix} \tag{3-10}$$

Substituting this displacement field into the governing equation (2-27) yields the following set of equations

$$\begin{bmatrix} Q^*_{11}-\lambda & Q^*_{12} & Q^*_{13} \\ Q^*_{12} & Q^*_{22}-\lambda & Q^*_{23} \\ Q^*_{13} & Q^*_{23} & Q^*_{33}-\lambda \end{bmatrix} \begin{bmatrix} E \\ F \\ G \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{3-11}$$

where Q^*_{11}, Q^*_{12}, Q^*_{22}, Q^*_{33} are expressed in the same form as Q_{11}, Q_{12}, Q_{22}, Q_{33} plus

$$Q^*_{13} = -k_2 \left[3B_{16} k_1^2 + B_{26} k_2^2\right]$$
$$Q^*_{23} = -k_1 \left[B_{16} k_1^2 + 3B_{26} k_2^2\right] \tag{3-12}$$

where $k_1 = \frac{m\pi}{a}$, $k_2 = \frac{n\pi}{b}$, $\lambda = \rho\,\Omega^2$. The determinant of the coefficient matrix in (3-11) must vanish for a non-trivial solution to yield the following frequency equation

$$\lambda^3 - H^*_1 \lambda^2 + H^*_2 \lambda - H^*_3 = 0 \tag{3-13}$$

where

$$H^*_1 = Q^*_{11} + Q^*_{22} + Q^*_{33}$$
$$H^*_2 = Q^*_{11}Q^*_{22} + Q^*_{11}Q^*_{33} + Q^*_{22}Q^*_{33} - Q^{*2}_{12} - Q^{*2}_{13} - Q^{*2}_{23} \tag{3-14}$$
$$H^*_3 = Q^*_{11}Q^*_{22}Q^*_{33} + 2Q^*_{12}Q^*_{13}Q^*_{23} - Q^*_{11}Q^{*2}_{23} - Q^*_{22}Q^{*2}_{13} - Q^*_{33}Q^{*2}_{12}$$

There will be three natural frequencies associated with three different modes, one predominantly transverse vibration mode and two predominantly inplane vibration modes, for given values of m and n. Although these three modes have the same half-wavelengths and nodal lines, the amplitude ratios between displacements of any mode will differ from those of the other two modes.

Numerical Results

Numerical results are obtained for antisymmetric cross-ply and angle-ply rectangular plates consisting of even number of identical orthotropic laminas, the fiber directions of any two adjacent layers are oriented at 0° and 90° to the x-axis of the plate for cross-ply laminate and + θ and - θ to the x-axis for angle-ply laminate. The elastic properties of individual layer are assumed that $E_\ell/E_t = 40$, $G_{\ell t}/E_t = 0.5$, and $\nu_{\ell t} = 0.25$. The half-wavelengths of any vibration mode in the x- and y-directions are denoted by ℓ_x and ℓ_y, and the three frequencies of the vibration mode are denoted by Ω_1, Ω_2, and Ω_3.

Comparisons between natural frequencies ω by neglecting inplane inertias and Ω_1 by including inplane inertias are listed in Table 1 for cross-ply plates and for angle-ply plates with orientation angle 45° for modes having the same half-wavelengths in x- and y-directions. The discrepancy between ω and Ω_1 is less than 1% even for the mode having half-wavelength to thickness ratios (ℓ_x/h and ℓ_y/h) as low as 10. In contrast to Ω_1 the two higher natural frequencies Ω_2 and Ω_3 are almost independent of the number of layers in an antisymmetric laminated plate. Figure 2 illustrates the percentage errors of ω compared to Ω_1 for modes having unequal half-wavelengths in the x- and y-directions for two-layer angle-ply plates. For a given half-wavelength in the x-direction, the percentage error raises as the half-wavelength in the y-direction increases. Usually the error due to neglecting inplane inertias in finding the natural frequency of predominantly flexural vibration is not significant for the lower modes of unsymmetric laminated rectangular thin plates. Table 2 shows the amplitude ratios associated with different vibration modes for both simply-supported cross-ply and angle-ply plates including inplane inertias. The lowest frequency for given half-wavelengths always occurs for a predominantly transverse vibration mode (the amplitude of the deflection is larger than the amplitudes of inplane displacements), and the other two higher natural frequencies are associated with the predominantly inplane vibration modes (one of the inplane-displacement amplitudes is larger than the deflection amplitude).

CHAPTER IV

SOME CLOSED FORM SOLUTIONS

Governing Equations Neglecting Inplane Inertias

For a thin plate, the inplane-inertia terms can be neglected if predominantly flexural vibrations are of interest (refer to Chapter III). Thus the governing equations for flexural vibration of an unsymmetric cross-ply rectangular plate, from Eq. (2-25), are given in the form

$$[L^c]\{u\}_o = \rho[m^*]\{\ddot{u}\}_o \tag{4-1}$$

and for an antisymmetric angle-ply plate, from Eq. (2-27), are given by

$$[L^a]\{u\}_o = \rho[m^*]\{\ddot{u}\}_o \tag{4-2}$$

where $[m^*]$ is a 3x3 matrix with all elements zero except $m^*_{33} = 1$. The displacement field for free harmonic vibration of a coupled laminated rectangular plate can be assumed in the form

$$\{u\}_o = \begin{bmatrix} u^o \\ v^o \\ w \end{bmatrix} = \begin{bmatrix} U(x,y)\ \sin\omega t \\ V(x,y)\ \sin\omega t \\ W(x,y)\ \sin\omega t \end{bmatrix} = \{U\}_o\ \sin\omega t \tag{4-3}$$

Use Eq. (4-3), Eq. (4-1) becomes

$$\left[(L^c_{ij} + m^*_{ij}\lambda)\right]\{U\}_o = 0 \tag{4-4}$$

and Eq. (4-2) becomes

$$[(L^a_{ij} + m^*_{ij}\lambda)]\,\{U\}_o = 0 \qquad (4\text{-}5)$$

where $\lambda = \rho\,\omega^2$ and the differential operators L^c_{ij}, L^a_{ij} $(i,\ j = 1,2,3)$ are defined in Chapter II.

Closed form solutions can be obtained for free vibration modes of both unsymmetric cross-ply and antisymmetric angle-ply rectangular plates neglecting inplane inertias with two opposite edges simply-supported in particular forms and the other two edges with general boundary conditions. These particular forms of simple supports, assumed at edges $y = 0$ and $y = b$, are prescribed as follows:

$$u^o = N_{yy} = w = M_{yy} = 0 \quad \text{(designated by S2)} \qquad (4\text{-}6)$$

for cross-ply plates, and

$$v^o = N_{xy} = w = M_{yy} = 0 \quad \text{(designated by S3)} \qquad (4\text{-}7)$$

for antisymmetric angle-ply plates

Method of Solution

By adopting the "Levy-type" solutions used by Forsberg [20] for free vibrations of circular cylindrical thin shells, the general solution for modal vibration of an unsymmetric cross-ply rectangular thin plate with two opposite edges $y = 0$ and $y = b$ simply-supported satisfying (4-6) can be assumed in the form

$$\begin{bmatrix} U \\ V \\ W \end{bmatrix} = \begin{bmatrix} \Phi_1(x) \sin \frac{n\pi y}{b} \\ \Phi_2(x) \cos \frac{n\pi y}{b} \\ \Phi_3(x) \sin \frac{n\pi y}{b} \end{bmatrix} \quad (n = 1, 2, \ldots\ldots) \qquad (4\text{-}8)$$

Substituting Eq. (4-8) into Eq. (4-4) yields the ordinary differential equations in the x-coordinate given by the following matrix form

$$[L^{c*}]\{\Phi\} = 0 \qquad (4\text{-}9)$$

where $[L^{c*}]$ is a 3x3 symmetric matrix, and L^{c*}_{ij} are defined by

$$L^{c*}_{11} = A_{11}(\ \ ),_{xx} - A_{66}\, k_2^2$$

$$L^{c*}_{12} = -(A_{12} + A_{66})\, k_2\, (\ \ ),_x$$

$$L^{c*}_{13} = -B_{11}(\ \ ),_{xxx} + (B_{12} + 2B_{66})\, k_2^2 (\ \ ),_x$$

$$L^{c*}_{22} = -A_{66}(\ \ ),_{xx} + A_{22}\, k_2^2$$

$$L^{c*}_{23} = (B_{12} + 2B_{66})\, k_2\, (\ \ ),_{xx} - B_{22}\, k_2^3$$

$$L^{c*}_{33} = D_{11}\, (\ \ ),_{xxxx} - 2(D_{12} + 2D_{66})\, k_2^2 (\ \ ),_{xx} + D_{22} k_2^4 - \lambda$$

where $k_2 = \frac{n\pi}{b}$.

It remains to determine $\{\Phi\}$ in such a form as to satisfy the equation (4-9) and the boundary conditions at $x = 0$ and $x = a$. Assume

$$\{\Phi\} = \{\bar{C}\}\, e^{\gamma x} \qquad (4\text{-}11)$$

and substitute this equation into (4-9), then a nontrivial solution yields an eighth degree characteristic equation

$$\Delta_1(\gamma^2) = P_o\gamma^8 + P_1\gamma^6 + P_2\gamma^4 + P_3\gamma^2 + P_4 = 0 \qquad (4\text{-}12)$$

where

$$P_o = A_{66}(B_{11}^2 - A_{11}D_{11}) \tag{4-13}$$

$$P_1 = k_2^2[D_{11}(A_{11}A_{22} - A_{12}^2 - 2A_{12}A_{66}) + 2A_{11}A_{66}(D_{12} + 2D_{66}) - A_{22}B_{11}^2 + 2A_{12}B_{11}(B_{12} + 2B_{66}) - A_{11}(B_{12} + 2B_{66})^2]$$

$$P_2 = k_2^4[A_{11}A_{66}(\lambda/k_2^4 - D_{22}) - A_{22}A_{66}D_{11} - 2(D_{12} + 2D_{66})(A_{11}A_{22} - A_{12}^2 - 2A_{12}A_{66}) - 2B_{11}B_{22}(A_{12} + A_{66}) - 2A_{12}(B_{12} + 2B_{66})^2 + 2(A_{11}B_{22} + A_{22}B_{11})(B_{12} + 2B_{66})]$$

$$P_3 = k_2^6[(D_{22} - \lambda/k_2^4)(A_{11}A_{22} - A_{12}^2 - 2A_{12}A_{66}) + 2A_{22}A_{66}(D_{12} + 2D_{66}) - A_{11}B_{22}^2 + 2A_{12}B_{22}(B_{12} + 2B_{66}) - A_{22}(B_{12} + 2B_{66})^2]$$

$$P_4 = k_2^8 A_{66}[(\lambda/k_2^4 - D_{22})A_{22} + B_{22}^2]$$

Similarly, the general solution for antisymmetric angle-ply rectangular plates with two opposite edges, $y = 0$ and $y = b$, simply-supported satisfying (4-7) can be assumed in the form

$$\begin{bmatrix} U \\ V \\ W \end{bmatrix} = \begin{bmatrix} \left(\sum_{k=1}^{8} E_k e^{\gamma_k x}\right) \cos \frac{n\pi y}{b} \\ \left(\sum_{k=1}^{8} F_k e^{\gamma_k x}\right) \sin \frac{n\pi y}{b} \\ \left(\sum_{k=1}^{8} G_k e^{\gamma_k x}\right) \sin \frac{n\pi y}{b} \end{bmatrix} \tag{4-14}$$

Substituting (4-14) into (4-5) and requiring a non-trivial solution, we can obtain the corresponding characteristic equation in the form (4-12) with coefficients P_r $(r = 0,1,2,3,4)$ defined as follows:

$$P_o = A_{11}(B^2_{16} - A_{66}D_{11}) \tag{4-15}$$

$$P_1 = k^2_2[D_{11}(A_{11}A_{22} - A^2_{12} - 2A_{12}A_{66}) + 2A_{11}A_{66}(D_{12} + 2D_{66})$$
$$-6A_{11}B_{16}B_{26} + 2(3A_{12} - 2A_{66})B^2_{16}]$$

$$P_2 = k^4_2[A_{11}A_{66}(\lambda/k^4_2 - D_{22}) - A_{22}A_{66}D_{11} - 2(D_{12} + 2D_{66})(A_{11}A_{22}$$
$$-A^2_{12} - 2A_{12}A_{66}) + 9A_{22}B^2_{16} + 9A_{11}B^2_{26} - 4(5A_{12} + 2A_{66})\ B_{16}B_{26}]$$

$$P_3 = k^6_2[(D_{22} - \lambda/k^4_2)(A_{11}A_{22} - A^2_{12} - 2A_{12}A_{66}) + 2A_{22}A_{66}(D_{12} + 2D_{66})$$
$$-6A_{22}B_{16}B_{26} + 2(3A_{12} - 2A_{66})\ B^2_{26}]$$

$$P_4 = k^8_2A_{22}[(\lambda/k^4_2 - D_{22})\ A_{66} + B^2_{26}]$$

The determination of natural frequencies will depart somewhat from the standard procedure in which is solved a characteristic equation with roots involving the natural frequency ω, and a determinant generated by the imposition of boundary conditions is required to vanish. Equation (4-9) will be satisfied by

$$\begin{bmatrix} \Phi_1(x) \\ \Phi_2(x) \\ \Phi_3(x) \end{bmatrix} = \begin{bmatrix} A \cos k_1\ (x - x_o) \\ B \sin k_1\ (x - x_o) \\ C \sin k_1\ (x - x_o) \end{bmatrix} \tag{4-16}$$

with proper relations between A, B, and C, where k_1 and x_o are some constants. Substituting (4-16) into (4-9), a non-trivial solution gives a formula for natural frequencies of cross-ply rectangular plates

$$\lambda = D_{11}k_1^4 + 2(D_{12} + 2D_{66})k_1^2 k_2^2 + D_{22}K_2^4 - \frac{1}{J_1}[k_1^2 (A_{66}k_1^2 + A_{22}k_2^2) J_2^2 - 2k_1^2 k_2^2 (A_{12} + A_{66})J_2J_3 + k_2^2 (A_{11}k_1^2 + A_{66}k_2^2) J_3^2] \quad (4\text{-}17)$$

where $$J_1 = (A_{11}k_1^2 + A_{66}k_2^2)(A_{66}k_1^2 + A_{22}k_2^2) - (A_{12} + A_{66})^2 k_1^2 k_2^2 \quad (4\text{-}18)$$

$$J_2 = B_{11}k_1^2 + (B_{12} + 2B_{66})\, k_2^2$$

$$J_3 = (B_{12} + 2B_{66})k_1^2 + B_{22}k_2^2$$

and $k_2 = \frac{n\pi}{b}$, k_1 is to be determined.

Similarly, for antisymmetric angle-ply plates, the following expressions

$$\begin{bmatrix} U \\ V \\ W \end{bmatrix} = \begin{bmatrix} E \sin k_1\ (x - x_o) \cos \frac{n\pi y}{b} \\ F \cos k_1\ (x - x_o) \sin \frac{n\pi y}{b} \\ G \sin k_1\ (x - x_o) \sin \frac{n\pi y}{b} \end{bmatrix} \quad (4\text{-}19)$$

with proper relations between E, F, and G will satisfy the governing equation (4-5) and the boundary conditions (4-7). In the same manner as that for cross-ply, a formula for natural frequencies of antisymmetric angle-ply rectangular plates can be obtained

$$\lambda = D_{11}k_1^4 + 2(D_{12} + 2D_{66})\, k_1^2 k_2^2 + D_{22}k_2^4 - \frac{1}{J_4}[k_1^2 (A_{11}k_1^2 + A_{66}k_2^2) J_5^2 - 2k_1^2 k_2^2 (A_{12} + A_{66})\, J_4J_5 + k_2^2 (A_{66}k_1^2 + A_{22}k_2^2) J_6^2 \quad (4\text{-}20)$$

where

$$J_5 = B_{16}k_1^2 + 3B_{26}k_2^2 \qquad (4\text{-}21)$$

$$J_6 = 3B_{16}k_1^2 + B_{26}k_2^2$$

and J_4 is the same form as J_1; again $k_2 = \frac{n\pi}{b}$.

Now, departing from the standard procedure, we will solve the characteristic equation $\Delta_1(\gamma^2) = 0$ such that the roots are functions of k_1 instead of the parameter ω. Putting the equation (4-17) or (4-20) into the equation (4-12) and detaching the roots $\gamma_{1,2} = \pm ik_1$, corresponding to the solution (4-16) or (4-19), the equation $\Delta_1(\gamma^2) = 0$ can be reduced to a polynomial of degree six

$$\Delta_1^*(\gamma^2) = -P_o^*\gamma^6 + P_1^*\gamma^4 - P_2^*\gamma^2 + P_3^* = 0 \qquad (4\text{-}22)$$

such that

$$\Delta_1(\gamma^2) = \Delta_1^*(\gamma^2)(\gamma^2 + k_1^2)$$

where

$$P_s^* = P_s^*(A_{ij}, B_{ij}, D_{ij}, \frac{n\pi}{b}, k_1), \quad (s = 0,1,2,3) \qquad (4\text{-}23)$$

A_{ij}, B_{ij} and D_{ij} are stiffness constants for laminated plates. Once the equation $\Delta_1^*(\gamma^2) = 0$ is solved the eigenfunctions (displacement functions) can be formed. If the equation $\Delta_1^*(\gamma^2) = 0$ has roots in the form

$$\gamma_k = \pm\alpha_2, \quad \pm(\alpha_3 \pm i\beta_3)$$

or $\qquad (k = 3, \ldots\ldots, 8) \qquad (4\text{-}24)$

$$\gamma_k = \pm\alpha_2, \quad \pm\alpha_3, \quad \pm\alpha_4$$

where α_2, α_3, α_4 and β_3 are to be real quantities regarding k_1 as the only unknown parameter, then the complete solutions (including the solution corresponding to the roots $\gamma_{1,2} = \pm ik_1$) of $\Phi_i(x)$ $(i = 1,2,3)$ for Eq. (4-9) can be written in the form

$$\Phi_3(x) = C_1 \sin k_1 x + C_2 \cos k_1 x + C_3 e^{-\alpha_2 x} + e^{-\alpha_3 x} (C_4 \sin\beta_3 x + C_5 \cos \beta_3 x) + C_6 e^{\alpha_2 x} + e^{\alpha_3 x} (C_7 \sin \beta_3 x + C_8 \cos\beta_3 x) \quad (4\text{-}25)$$

or

$$\Phi_3(x) = C_1 \sin k_1 x + C_2 \cos k_1 x + C_3 e^{-\alpha_2 x} + C_4 e^{-\alpha_3 x} + C_5 e^{-\alpha_4 x} + C_6 e^{\alpha_2 x} + C_7 e^{\alpha_3 x} + C_8 e^{\alpha_4 x}$$

with similar expressions for $\Phi_1(x)$ and $\Phi_2(x)$. It is noted that all coefficients in (4-25) are real constants. The coefficients in $\Phi_1(x)$ and $\Phi_2(x)$ can be expressed in terms of C_k $(k = 1,\ldots.8)$ by means of the coupled governing equations given in Eq. (4-9). Therefore, there are eight unknown constants C_k $(k = 1,\ldots,8)$ remaining. The ratios between these constants can be determined from the boundary conditions at $x = 0$ and $x = a$.

If we introduce a new coordinate x^* such that $x^* = a - x$, the complete solutions, $\Phi_i^*(x^*)$ $(i = 1,2,3)$, for Eq. (4-9) are expressed in the same form as $\Phi_i(x)$, where x is replaced by x^* and C_k $(k = 1,\ldots,8)$ by C_k^*. Now, the ratios between these constants C_k^* are determined from the boundary conditions at $x^* = 0$ and $x^* = a$ in the new coordinate. For any given vibration mode the displacements expressed by both coordinates x and x^* are the same at every point, i.e. $\Phi_i(x) = \Phi_i^*(a - x)$. Using the equation $\Phi_3(x) = \Phi_3^*(a - x)$ and equating the corresponding terms on both sides, we find the relation

$$C_1 \sin k_1 x + C_2 \cos k_1 x = C_1^* \sin k_1 (a - x) + C_2^* \cos k_1 (a - x) \qquad (4\text{-}26)$$

which yields two equations

$$\begin{aligned} C_1 &= C_2^* \sin k_1 a - C_1^* \cos k_1 a \\ C_2 &= C_1^* \sin k_1 a + C_2^* \cos k_1 a \end{aligned} \qquad (4\text{-}27)$$

Solving $\sin k_1 a$ and $\cos k_1 a$ from (4-27), we can find the relation

$$\tan k_1 a = \frac{C_2 C_1^* + C_1 C_2^*}{C_2 C_2^* - C_1 C_1^*} \qquad (4\text{-}28)$$

Thus the formula for determining k_1 is obtained

$$\begin{aligned} k_1 a &= \arctan \frac{-C_2/C_1 - C_2^*/C_1^*}{1 - (-C_2/C_1)(-C_2^*/C_1^*)} + m\pi \qquad (4\text{-}29) \\ &= \arctan\left(-\frac{C_2}{C_1}\right) + \arctan\left(-\frac{C_2^*}{C_1^*}\right) + m\pi, \quad (m = 0, 1, 2, ..) \end{aligned}$$

We must take for the arctangent its principal value. It is noted that the numbers of nodal lines running "perpendicular" to the x- and y-directions, s_x and s_y, for a given mode are $s_x = m + 1$ and $s_y = n + 1$ respectively.

If the boundary conditions at $x = 0$ and $x = a$ are the same, formula (4-29) becomes

$$k_1 a = 2 \arctan\left(-\frac{C_2}{C_1}\right) + m\pi, \quad (m = 0, 1, ...) \qquad (4\text{-}30)$$

Here, odd values of m correspond to symmetric modes (about the middle line $x = a/2$), even values of m correspond to antisymmetric modes.

If the edge $x = a$ is simply-supported with case S2 for cross-ply plates and S3 for antisymmetric angle-ply plates, formula (4-29) becomes

$$k_1 a = \arc\tan\left(-\frac{C_2}{C_1}\right) + m\pi, \quad (m = 0,1,\ldots) \tag{4-31}$$

The eight boundary conditions, four at each edge, lead directly to eight homogeneous equations for the eight unknown constants C_k $(k = 1,\ldots,8)$ which are linearly dependent. Since the roots of the characteristic equation $\Delta_1(\gamma^2) = 0$ are functions of k_1, i.e. k_1 is involved as a parameter in the displacement functions, hence from any seven equations of the eight homogeneous equations satisfying the boundary conditions the ratios C_k/C_1 $(k = 2,\ldots,8)$ can be solved as functions of k_1. As a result, k_1 can be solved from the formula (4-29) for any vibration mode. Once the value of k_1 is found the corresponding natural frequency and mode shape can be determined from the frequency formula (4-17) (or (4-20)) and the displacement functions, respectively. If the boundary conditions at $x = 0$ and $x = a$ are the same, the number of unknown constants C_k can be reduced from 8 to 5 (C_1 and C_2 not changed) by using the properties of symmetric and antisymmetric mode shapes. Therefore, only four boundary conditions are needed to solve the ratios C_k/C_1 (now $k = 2,\ldots,5$).

The procedure of numerical computation for the solution is taken as follows. We select a given laminated plate (i.e., material properties, type of layered arrangement and plate dimensions), an assumed number of n, and a specific set of boundary conditions at $x = 0$ and $x = a$. Using an iteration procedure, the formula (4-29) can be considered as the form $(k_1)_{i+1} = f((k_1)_i)$. Starting from some initial estimate for k_1 which can be easily chosen for any mode, we can solve the equation $\Delta_1^*(\gamma^2) = 0$. Substituting these values obtained plus the value of k_1 (i.e. all roots of the equation $\Delta_1(\gamma^2) = 0$) into any seven of the eight homogeneous equations satisfying the boundary conditions (the resulting matrix is 7x8), the ratio C_2/C_1 can be obtained and the resulting value $f((k_1)_o) = (k_1)_1$ is used for the next iteration. To repeat this iteration, the value of k_1 can be obtained at the desired accuracy and thus the natural frequency can be obtained by the frequency formula. The iteration in the present procedure will converge very fast in contrast to the standard procedure (using trial and error to find ω making the determinant zero) and the characteristic equation is reduced to a lower degree of equation (i.e., reduce from eighth degree to six degree by detaching the two purely imaginary roots). Therefore, the computation time by the present procedure will be much less compared with the standard procedure for the natural frequency at the same order of accuracy. Particularly, the present procedure is much more convenient than the standard procedure for finding higher modes.

In view of the fact that there will be no purely imaginary roots for the characteristic equation of the lowest frequency with a given value of n for a homogeneous isotropic rectangular plate with edges $x = 0$ and $x = a$ free, we must also consider this possibility for laminated plates. If there is no purely imaginary roots for the equation (4-12), the roots of the equation (4-12) are to be the form

$$\gamma_k = \pm \alpha_1,\ \pm \alpha_2,\ \pm (\alpha_3 \pm i\ \beta_3) \qquad (4\text{-}32)$$

or

$$\gamma_k = \pm \alpha_1,\ \pm \alpha_2,\ \pm \alpha_3,\ \pm \alpha_4 \qquad (k = 1, \ldots, 8)$$

$$\gamma_k = \pm (\alpha_1 \pm i\beta_1),\ \pm (\alpha_2 \pm i\beta_2)$$

If the roots are in the first or the second form, then the corresponding transverse deflection in x coordinate is to be the form (symmetric)

$$\Phi_3(x) = C_1 \sinh \alpha_1 x + C_2 \cosh \alpha_1 x + C_3[e^{-\alpha_2 x} + e^{-\alpha_2(a-x)}] \qquad (4.33)$$

$$+ C_4 [e^{-\alpha_3 x} \sin \beta_3 x + e^{-\alpha_3(a-x)} \sin \beta_3(a-x)]$$

$$+ C_5[e^{-\alpha_3 x} \cos \beta_3 x + e^{-\alpha_3(a-x)} \cos \beta_3(a-x)]$$

$$\Phi_3(x) = C_1 \sinh \alpha_1 x + C_2 \cosh \alpha_1 x + C_3[e^{-\alpha_2 x} + e^{-\alpha_2(a-x)}]$$

$$+ C_4[e^{-\alpha_3 x} + + e^{-\alpha_3(a-x)}] + C_5[e^{-\alpha_4 x} + e^{-\alpha_4(a-x)}]$$

Thus, from the symmetric property $\Phi_3(x) = \Phi_3(a - x)$, the formula for the determination of α_1 is

$$\tanh \frac{\alpha_1 a}{2} = - \frac{C_1}{C_2} \qquad (4\text{-}34)$$

and the natural frequency will be determined by the same formula as (4-17) or (4-20), where k_1^2 is replaced by $-\alpha_1^2$. If the roots are in the third form (there is not such case in the problems solved), the present method is not valid but the standard procedure can be applied.

Numerical Results

Frequency computations by the present procedure and the standard procedure are compared for the fundamental modes of a two-layer anti-symmetric angle-ply square plates with edges x = 0 and x = a rigidly-clamped for nine different orientations (material properties $E_\ell/E_t = 40$, $G_{\ell t}/E_t = 0.5$, $\nu_{\ell t} = 0.25$). The differences of two successive iterations are required to be not greater than 0.00001 and 0.0001 for convergence to k_1a and $\omega b^2(\rho/E_t\ h^3)^{1/2}$ in the two procedures, respectively. All natural frequencies obtained by two procedures coincide to at least five figures and the computation time by the present procedure is less than one fourth of that by the standard procedure.

Natural frequencies are evaluated for both antisymmetric cross-ply and angle-ply plates consisting of even number of identical orthotropic layers. Frequency as a function of E_ℓ/E_t for mode having two nodal lines (i.e., $s_x = x_y = 2$) running "perpendicular" to the x- and y-directions for an antisymmetric cross-ply square plate with two cases of boundary conditions is illustrated in Figure 3. Frequency as a function of angle-ply orientation (θ) for similar mode and boundary conditions is shown in Figure 4 for an antisymmetric angle-ply square plate (material properties $E_\ell/E_t = 40$, $G_{\ell t}/E_t = 0.5$, $\nu_{\ell t} = 0.25$). The effect of bending-stretching coupling on natural frequency of antisymmetric laminated plate depends on the number of layers and the ratio E_ℓ/E_t; furthermore, for an angle-ply plate this effect also depends on the ply-orientation.

The effect of coupling between bending and stretching described by $(\omega_d - \omega_L)/\omega_d$, where ω_d denotes the natural frequency for the "decoupled" plate (all B_{ij} are zero) and ω_L denotes the natural frequency of an L-layer laminate, as a function of ply-orientation is illustrated in Figure 5 for an antisymmetric angle-ply square plate for various boundary conditions (material properties $E_\ell/E_t = 40$, $G_{\ell t}/E_t = 0.5$, $\nu_{\ell t} = 0.25$). It can be seen that the influence of boundary conditions on the coupling effect is generally not significant. However, as might be expected, there are significant differences between results obtained for free edges and those obtained for supported edges.

The effects of inplane boundary conditions on natural frequency are investigated for certain types of square laminated plates (material properties $E_\ell/E_t = 40$, $G_{\ell t}/E_t = 0.5$, $\nu_{\ell t} = 0.25$) for three cases (C1, C2, and C3) of clamped edges at $x = 0$ and $x = a$, the fundamental natural frequencies of these plates are shown in Table 3. Obviously, the effect of inplane boundary conditions when edges are clamped is very small for cross-ply plates. For angle-ply plates, the effect of inplane boundary conditions on the laminate response can be affected by the number of layers and the angle-ply arrangement. It is interesting to note that the frequency of a four-layer angle-ply square plate is lower for the plate with plies oriented at 45°/-60°/60°/-45° than for the plate with plies oriented at either 45°/-45°/45°/-45° or 60°/-60°/60°/-60°, and that the effect of inplane boundary conditions is much greater for the first kind of arrangement than for either one of the second and the third arrangements as shown in Table 3.

For a homogeneous rectangular plate with two opposite free edges, a common approximation to some of the modes is a pattern corresponding to cylindrical bending. A comparison between exact solution (ω_E) and cylindrical bending solution (ω_C) for natural frequencies of homogeneous orthotropic plates with two-opposite edges $y = 0$ and $y = b$ simply-supported and the other two edges free is shown in Table 4 for different material properties and various half-wavelength to width ratios ℓ_y/a. The solution (ω_B) obtained by the "beam" equation is also included for comparison, where the Young's modulus has been replaced by by the rigidity constant H which involves Poisson's ratio. The error of cylindrical bending solution is not significant for small value of Poisson's ratio. However, the cylindrical bending solution for natural frequencies may lead to a great error for coupled laminated plates with high ratios E_{ℓ}/E_t. A comparison between exact solutions and cylindrical bending solutions for natural frequencies as functions of ply-orientation is illustrated in Figure 6 for an antisymmetric angle-ply plate (material properties $E_{\ell}/E_t = 40$, $G_{\ell t}/E_t = 0.5$, $\nu_{\ell t} = 0.25$). The error produced by the cylindrical bending solution for a four-layer angle-ply plate with $\theta = 45°$ is about 15% at $\ell_y/a = 1.0$, at $\ell_y/a = 2.0$ this error increases to about 28% high. Natural frequencies as functions of the aspect ratio for different modes are shown in Figure 7 for a two-layer cross-ply plate with two opposite edges x = 0 and x = a free, ($E_{\ell}/E_t = 40$, $G_{\ell t}/E_t = 0.5$, $\nu_{\ell t} = 0.25$). Obviously the lowest natural frequency of a fixed value n is almost independent of the aspect ratio, therefore, the cylindrical bending solution can be applied without significant error for this type of plate.

The roots of the characteristic equation $\Delta_1(\gamma^2) = 0$ and the ratios of the coefficients of exponential terms to the amplitude of the harmonic term, $C_1 \sin k_1 x + C_2 \cos k_1 x = C \sin k_1(x - x_0)$, are listed in Table 5 for a two-layer cross-ply square plate for two cases of edge conditions. Similar results are shown in Table 6 for a two-layer 45°/-45° angle-ply square plate with two cases of boundary conditions. In general, the decay constants for exponential terms are much greater than 1 or the coefficients of these exponential terms are negligibly small compared with C —— the amplitude of the harmonic term. Usually, the decay constants for exponential terms increase for higher modes.

Modal Stresses

Stress resultants and stress couples for unsymmetric laminated plates can be determined from the constitutive equation (2-11) once the displacement functions are given. Normal stress resultant N_{xx} and bending moment M_{xx} along rigidly-clamped edges $x = 0$ and $x = a$, which are of most interest, and along middle line $x = a/2$ are obtained for both antisymmetric cross-ply and angle-ply plates consisting of even number of identical orthotropic layers (having properties $E_\ell/E_t = 40$, $G_{\ell t}/E_t = 0.5$, $\nu_{\ell t} = 0.25$). The nondimensional normal stress resultant N^*_{xx} and bending moment M^*_{xx} are introduced such that

$$\left.\begin{aligned} N_{xx} &= N^*_{xx} \left(\frac{E_t h^2 C}{b^2} \right) \sin \frac{n\pi y}{b} \\ M_{xx} &= M^*_{xx} \left(\frac{E_t h^3 C}{b^2} \right) \sin \frac{n\pi y}{b} \end{aligned}\right\} \text{for cross-ply plate} \qquad (4\text{-}35)$$

and

$$\left.\begin{aligned} N_{xx} &= N^*_{xx} \left(\frac{E_t h^2 C}{b^2} \right) \cos \frac{n\pi y}{b} \\ M_{xx} &= M^*_{xx} \left(\frac{E_t h^3 C}{b^2} \right) \sin \frac{n\pi y}{b} \end{aligned}\right\} \text{for angle-ply plate} \qquad (4\text{-}36)$$

where C is the maximum transverse displacement. Also, N^*_{xo}, M^*_{xo} and N^*_{xm}, M^*_{xm} are designated for the nondimensional normal stress resultant and bending moment along rigidly-clamped edge $x = 0$ and along middle line $x = a/2$, respectively.

Results for different modes of a two-layer cross-ply square plate with edges $y = 0$, b simply-supported S2 and edges $x = 0$, a rigidly-clamped are listed in Table 7. Similar results for two types of two-layer angle-ply square plate, plies oriented at $45^\circ/-45^\circ$ and $65^\circ/-65^\circ$, are listed in Table 8. Results clearly show that the magnitude of the bending moment is higher for higher mode, i.e. the magnitude of the bending moment increases as the halfwave length decreases. This is in accord with the trend of natural frequencies listed in Table 7 and 9. However, this trend is not found for the normal stress resultant. Normal stress resultants and bending moments as functions of the ply-orientation along rigidly-clamped edges $x = 0$ and $x = a$ are shown in Figure 8 for antisymmetric angle-ply square plates consisting of different numbers of identical orthotropic layers. The coupling effect between bending and stretching reduces the bending moment while increasing the normal stress resultant as compared to the decoupled solution (all B_{ij} are zero). That is bending moment increases while normal stress resultant decreases as the number of layers in the laminated plate increases. Furthermore, it can be seen that normal stress resultant is almost inversely proportional to the number of layers in the plate and becomes zero for a "decoupled" plate.

CHAPTER V

ASYMPTOTIC SOLUTIONS FOR FREE VIBRATIONS OF RECTANGULAR PLATES

Introduction

An asymptotic method based on the concept of a dynamic edge effect was developed by Bolotin [13] for finding the eigenvalues and eigenfunctions for one class of homogeneous linear boundary value problems $L(\varphi) - \lambda M(\varphi) = 0$ in a rectangular region, where L and M are the linear differential operators with constant coefficients of order 2n and 2(n-1) respectively, not containing odd order derivatives. According to this method the asymptotic solution for eigenfunctions is expressed as a sum of a generating (or interior) solution and a corrective solution which has been called the "dynamic edge effect." The generating solution, being a product of trigonometric functions, satisfies the governing equation and yields an algebraic equation for determining eigenvalues as function of "wave numbers."

The generating solution will, in general, not satisfy the boundary conditions. Tests show that thin-walled elements fracture from fatigue most frequently near clamped edges [16]. Hence it is important to represent eigenfunctions as accurately as possible near the edges of a plate. For each boundary, we can construct an asymptotic solution satisfying the governing equation and the conditions at the relevant boundary. The number of these solutions is equal to the number of boundaries. Thus the asymptotic solution for an eigenfunction for the entire region, except in the neighborhood of corners, is found by joining these solutions. All these solutions tend to the generating solution toward the internal region for a non-degenerate dynamic edge effect (i.e. the corrective solution can be neglected in the internal region). By matching these solutions, we can obtain a set of transcendental equations for the wave numbers. Then an approximation to an eigenvalue can be obtained. The asymptotic solution obtained is valid for all cases except the cases for which the eigenfunction does not tend to the generating solution toward the internal region (a degenerate dynamic edge effect).

This asymptotic method has been applied successfully to the determinations of natural frequencies and natural modes for homogeneous isotropic rectangular plates having simply-supported and clamped edges [13,14,15,16], and for a homogeneous orthotropic rectangular clamped-plate [14]. Using this method, Bolotin also studied the edge effect in the free vibrations of isotropic-thin shells [13,16] and the natural plane vibrations of a homogenous isotropic rectangular parallelepiped [17]. In addition, on the basis of the asymptotic method, Bolotin investigated the density of natural frequencies of homogeneous isotropic plates and shells [21] as well as the problems of broadband random vibrations of elastic systems (beams, plates and shells) [22].

We will now extend the application of the Bolotin asymptotic method to the analysis of free vibrations for unsymmetric laminated rectangular plates with general boundary conditions. For the purpose of a reliable comparison between results obtained by the present asymptotic method and by the other approximate methods, natural frequencies of homogeneous orthotropic rectangular plates with a wide variety of boundary conditions are also included.

Analysis of Decoupled Laminated Plates and Homogeneous Orthotropic Plates

For a symmetric laminate (all B_{ij} are zero) with twist-coupling constants D_{16} and D_{26} zero or for an antisymmetric alternating-layered plate when the total number of identical orthotropic layers becomes so large that all the stiffness components B_{ij} are negligibly small, the governing equation of free transverse vibration is in decoupled form and is given by

$$D_{11}W,_{xxxx} + 2(D_{12} + 2D_{66})\, W,_{xxyy} + D_{22}W,_{yyyy} - \lambda W = 0 \qquad (5\text{-}1)$$

where $\lambda = \rho\omega^2$. If we replace D_{11} by D_x, D_{22} by D_y, D_{12} by $\nu_{yx}D_x$, and D_{66} by D_{xy}, Eq. (5-1) becomes the familiar governing equation for flexural vibration of a homogeneous orthotropic rectangular plate, i.e.

$$D_x W,_{xxxx} + 2\,H\,W,_{xxyy} + D_y W,_{yyyy} - \lambda\,W = 0 \qquad (5\text{-}2)$$

where $H = \nu_{yx}D_x + 2\,D_{xy}$.

Eqs. (5-1) and (5-2) will be satisfied by

$$W = C \sin k_1(x - x_o) \sin k_2(y - y_o) \qquad (5\text{-}3)$$

where k_1, k_2, x_o, and y_o are certain constants. Constants $k_1 = \pi/\ell_x$ and $k_2 = \pi/\ell_y$ are called wave numbers, where ℓ_x and ℓ_y are half-wave-lengths in the x- and y-directions. Substituting Eq. (5-3) into (5-1), a formula for the natural frequency as a function of k_1 and k_2 is obtained:

$$\lambda = D_{11} k_1^{\,4} + 2(D_{12} + 2D_{66})k_1^{\,2} k_2^{\,2} + D_{66} k_2^{\,4} \qquad (5\text{-}4)$$

In general, the expression (5-3) will not satisfy the boundary conditions. We may try to find an asymptotic solution for the natural mode for the entire region except in the neighborhood of corners by considering the expression (5-3) as the generating solution and adding some corrective solution to satisfy the conditions at the relevant boundary.

Now consider the boundary $x = 0$; the solution for the deflection near this edge will be sought in the form

$$W = \bar{C} \; e^{\gamma x} \sin k_2(y - y_o) \qquad (5\text{-}5)$$

The substitution of (5-5) and (5-4) into (5-1) yields

$$D_{11}\gamma^4 - 2(D_{12} + 2D_{66})k_2^{\,2}\gamma^2 - [D_{11}k_1^{\,2} + 2(D_{12} + 2D_{66})k_2^{\,2}]k_1^2 = 0$$

i.e.

$$(\gamma^2 + k_1^{\,2})\left\{D_{11}\gamma^2 - [D_{11}k_1^{\,2} + 2(D_{12} + 2D_{66})k_2^{\,2}]\right\} = 0 \qquad (5\text{-}6)$$

Obviously, the equation (5-6) has the roots $\pm\, ik_1$, $\pm\, [k_1^2 + \frac{2(D_{12} + 2D_{66})}{D_{11}} k_2^2]^{1/2}$.
Discarding the positive real root, the deflection near the edge $x = 0$ can be expressed in the form

$$W(x|0) = (C_1 \sin k_1 x + C_2 \cos k_1 x + C_3 e^{-\alpha_2 x}) \sin k_2(y - y_o) \qquad (5\text{-}7)$$

where $\alpha_2 = [k_1^2 + \frac{2(D_{12} + 2D_{66})}{D_{11}} k_2^2]^{1/2}$, and all coefficients are real constants. The sum of the first and second terms in (5-7) corresponds to the generating solution (5-3), and the third term is the corrective solution which has been called the dynamic edge effect by Bolotin. The ratios between constants C_1, C_2, and C_3 can be determined from the two boundary conditions at $x = 0$.

Similarly, the deflection near the edge $x = a$ can be written in the form

$$W(x|a) = [C_1^* \sin k_1(a-x) + C_2^* \cos k_1(a-x) + C_3^* e^{-\alpha_2(a-x)}] \cdot \sin k_2 (y-y_o) \quad (5\text{-}8)$$

where the ratios between constants C_1^*, C_2^* and C_3^* are to be determined from the two boundary conditions at $x = a$.

In the same manner, the asymptotic solution for the deflection near the edge $y = 0$ is

$$W(y|0) = (D_1 \sin k_2 y + D_2 \cos k_2 y + D_3 e^{-\alpha_2' y}) \sin k_1(x - x_o) \quad (5\text{-}9)$$

where $\alpha_2' = [k_2^2 + \frac{2(D_{12} + 2D_{66})}{D_{22}} k_1^2]^{1/2}$, and the deflection near the edge $y = b$ is

$$W(y|b) = [D_1^* \sin k_2(b-y) + D_2^* \cos k_2(b-y) + D_3^* e^{-\alpha_2'(b-y)}] \sin k_1 (x-x_o) \quad (5\text{-}10)$$

The corrective solutions will decay rapidly with increasing distance from each boundary. Thus by matching the solutions $W(x|o)$ and $W(x|a)$ in the internal region and neglecting the corrective solutions, we can obtain the following relation

$$C_1 \sin k_1 x + C_2 \cos k_1 x = C_1^* \sin k_1(a-x) + C_2^* \cos k_1(a-x) \quad (5\text{-}11)$$

which is the same form as (4-26). The formula for finding the wave number k_1 is

$$k_1 a = \text{arc tan}\,(-\frac{C_2}{C_1}) + \text{arc tan}\,(-\frac{C_2^*}{C_1^*}) + m\pi, \quad (m=0,1,\ldots) \qquad (5\text{-}12)$$

Similarly

$$k_2 b = \text{arc tan}\,(-\frac{D_2}{D_1}) + \text{arc tan}\,(-\frac{D_2^*}{D_1^*}) + n\pi, \quad (n=0,1,\ldots) \qquad (5\text{-}13)$$

where the arctangent must take its principal value. The numbers of nodal lines running "perpendicular" to the x- and y-directions for a natural mode with fixed wave numbers k_1 and k_2 are given by $s_x = m + 1$ and $s_y = n + 1$, respectively.

If the boundary condition at $x = 0$ and $x = a$ are the same, Eq. (5-12) becomes

$$k_1 a = 2\,\text{arctan}\,(-\frac{C_2}{C_1}) + m\pi, \quad (m = 0,1,\ldots) \qquad (5\text{-}14)$$

Here, odd numbers of m correspond to symmetric (about the middle line $x = \frac{a}{2}$), and even numbers of m correspond to antisymmetric modes.

If the edge $x = a$ is simply-supported, then $C_2^*/C_1^* = 0$ and Eq. (5-12) becomes

$$k_1 a = \text{arctan}\,(-\frac{C_2}{C_1}) + m\pi, \quad (m = 0,1,\ldots) \qquad (5\text{-}15)$$

If all four edges are simply-supported then $k_1 a = m\pi$ and $k_2 = n\pi$ the solution becomes exact.

If a decoupled laminated plate or a homogeneous orthotropic plate is clamped at $x = 0$, the ratios between constants can be obtained:

$$-\frac{C_2}{C_1} = \frac{C_3}{C_1} = k_1 / \left[k_1^{\,2} + \frac{2(D_{12} + 2D_{66})}{D_{11}} k_2^{\,2}\right]^{1/2} \tag{5-16}$$

If the plate is free at x = 0, then

$$-\frac{C_2}{C_1} = \left[k_1^2 + \frac{(D_{12} + 4D_{66})}{D_{11}} k_2^2\right]^2 k_1 / (k_1^2 + \frac{D_{12}}{D_{11}} k_2^2)^2 \left[k_1^2 + \frac{2(D_{12} + 2D_{66})}{D_{11}} k_2^2\right]^{1/2}$$

(5-17)

$$-\frac{C_3}{C_1} = \left[k_1^2 + \frac{(D_{12} + 4D_{66})}{D_{11}} k_2^2\right] k_1 / (k_1^2 + \frac{D_{12}}{D_{11}} k_2^2) \left[k_1^2 + \frac{2(D_{12} + 2D_{66})}{D_{11}} k_2^2\right]^{1/2}$$

For a plate with given material properties, plate dimensions and specific boundary conditions at all edges, the wave numbers k_1 and k_2 (the only two unknowns in (5-12) and (5-13)) thus can be solved from Eqs. (5-12) and (5-13) for any natural mode. Once the wave numbers are obtained the corresponding natural frequency can be determined from Eq . (5-4).

From Eqs. (5-14) and (5-17), it can be seen that the value $k_1 = 0$ is also a solution (there is a solution other than 0) to the equation (5-14) with m = 0 for a plate with edges x = 0 and x = a free. Therefore, in this case, the plate vibration is assumed to behave as if in cylindrical bending for the lowest mode for a given wave number k_2 in the y-direction.

If we let $\mu = k_1/k_2$ ($k_2 \neq 0$) and R = b/a, then Eqs. (5-12) and (5-13) can be combined into a single equation as follows:

$$\mu = R \, \frac{\arctan(-C_2/C_1) + \arctan(-C_2^*/C_1^*) + m\pi}{\arctan(-D_2/D_1) + \arctan(-D_2^*/D_1^*) + n\pi}, \quad (m, n = 0, 1, \ldots) \tag{5-18}$$

where C_2/C_1, C_2^*/C_1^*, D_2/D_1, and D_2^*/D_1^* are functions of μ only such that

$$-\frac{C_2}{C_1} = \mu / \left[\mu^2 + \frac{2(D_{12} + 2D_{66})}{D_{11}}\right]^{1/2} \tag{5-19}$$

for the clamped edge at $x = 0$, and

$$-\frac{C_2}{C_1} = \left[\mu^2 + \frac{D_{12} + 4D_{66}}{D_{11}}\right]^2 \mu / \left(\mu^2 + \frac{D_{12}}{D_{11}}\right)^2 \left[\mu^2 + \frac{2(D_{12} + 2D_{66})}{D_{11}}\right]^{1/2} \tag{5-20}$$

for the free edge at $x = 0$.

For a square plate with rigidities D_{11} and D_{22} equal and boundary conditions at two pairs of opposite edges are similar, the wave numbers k_1 and k_2 will be equal (i.e. $\mu = 1$) for those modes having same nodal lines running along both x- and y-directions. In this case the wave numbers and natural frequency can be evaluated by hand. As an example, in the case of a plate clamped at all edges, the wave numbers k_1 and k_2 can be obtained from the following formula

$$k_1 a = k_2 a = 2 \arctan \left[1/\sqrt{1 + \frac{2(D_{12} + 2D_{66})}{D_{11}}}\right] + m\pi, \quad (m=1,2,\ldots) \tag{5-21}$$

The numerical computation in this asymptotic procedure is very simple. Using iteration procedure, Eq. (5-18) can be considered as $\mu_{i+1} = f(\mu_i)$. We can easily choose a proper initial value of μ for any natural mode of a plate. Substituting this initial value μ_o into the right hand side of (5-18), we can obtain a new value $\mu_1 = f(\mu_o)$. We repeat the iteration, which will converge rapidly, until a value μ which has a desired accuracy is obtained. Once μ is found the wave numbers k_1 and k_2 can be obtained from Eqs. (5-12) and (5-13), respectively. With these wave numbers, K_1 and K_2, an estimate for the corresponding natural frequency can be determined from Eq. (5-4).

Natural frequencies of decoupled laminated plates and of homogeneous orthotropic plates are obtained for certain types of boundary conditions. Results for decoupled laminated plates will be discussed in the next two sections together with those for plates with coupling. Natural frequencies for homogeneous orthotropic plates are tabulated in Tables 10 and 11 in comparison with those values obtained by other investigators: Kanazawa and Kawai [23] used an integral equation formulation, Dickinson [24] used the Fourier series method, Durvasula and Srinivasan [25] and Young [26] applied the Ritz method, Gontkevich [27] used the Southwell method to obtain lower bounds for natural frequencies of cantilever plates, and Bazley, Fox, and Stadter [28] obtained the upper and lower bounds for frequencies of cantilever plates. Results show that values obtained by the asymptotic method are acceptable for all modes except the modes having half-wavelengths in both directions simultaneously longer than the shorter side of the plate. For these lower modes the dynamic edge effects decay slowly, thus the rejection of non-decay term may produce great error. The asymptotic solution is more accurate for higher mode in contrast to other methods. It appears that the asymptotic solution gives a lower bound for natural frequencies of rectangular plates with edges supported.

Analysis of Unsymmetric Cross-Ply Plates

For an unsymmetric cross-ply rectangular plate satisfying the basic assumptions discussed in Chapter II, the governing equations of vibration neglecting inplane inertias are given in Eq. (4-4) and the boundary conditions can be any combination of Eqs. (2-29) and (2-30).

Separation of variables is possible for the governing equations, therefore, the asymptotic method is applicable.

Eq. (4-4) will be satisfied by assuming

$$\begin{bmatrix} U \\ V \\ W \end{bmatrix} = \begin{bmatrix} A \cos k_1(x - x_o)\sin k_2(y - y_o) \\ B \sin k_1(x - x_o)\cos k_2(y - y_o) \\ C \sin k_1(x - x_o)\sin k_2(y - y_o) \end{bmatrix} \tag{5-22}$$

Putting (5-22) into (4-4), a non-trivial solution leads to an algebraic equation for the natural frequency

$$\lambda = D_{11}k_1^{\,4} + 2(D_{12} + 2D_{66})k_1^{\,2}k_2^{\,2} + D_{22}k_2^{\,4} - \frac{1}{J_1}\,[(A_{66}k_1^{\,2} + A_{22}k_2^{\,2})J_2^2k_1^{\,2} - 2(A_{12} + A_{66})J_2J_3k_1^{\,2}k_2^{\,2} + (A_{11}k_1^{\,2} + A_{66}k_2^{\,2})\,J_3^{\,2}k_2^{\,2}] \tag{5-23}$$

where J_1, J_2, and J_3 are defined in (4-18). In general, the solution (5-22) will not satisfy the boundary conditions unless all edges are subjected to simple supports S2; this solution is called the generating solution. In order to satisfy the boundary conditions at the relevant boundary, it is necessary to add some corrective solutions to the generating solution to form the asymptotic solution for the natural mode near the boundary considered.

Considering the boundary x = 0, the solution for the natural mode near this edge will be sought in the form

$$\begin{bmatrix} U \\ V \\ W \end{bmatrix} = \begin{bmatrix} \Phi_1(x)\ \sin k_2\ (y-y_o) \\ \Phi_2(x)\ \cos k_2\ (y-y_o) \\ \Phi_3(x)\ \sin k_2\ (y-y_o) \end{bmatrix} \tag{5-24}$$

When a displacement field of this form is substituted into the governing equation (4-4), the variables separate leaving a system of ordinary differential equations in the x-coordinate as Eq. (4-9). Unknown functions $\Phi_i(x)$ $(i = 1, 2, 3)$ are to be determined to satisfy Eq. (4-9) and the boundary conditions only at edge $x = 0$. In the similar procedure as that for closed form solution discussed in Chapter IV, assume $\Phi_i(x) = \bar{C}_i e^{\gamma x}$ $(i = 1, 2, 3)$ and substitute into Eq. (4-9), a nontrivial solution leads to an eighth degree characteristic equation

$$\Delta_1(\gamma^2) = P_o\gamma^8 + P_1\gamma^6 + P_2\gamma^4 + P_3\gamma^2 + P_4 = 0 \qquad (5\text{-}25)$$

where coefficients P_r $(r = 0, 1, 2, 3, 4)$ are defined in Eq. (4-13). Substituting Eq. (5-23) for λ into these coefficients, then all coefficients will be functions of the wave numbers k_1 and k_2 (λ is not involved), i.e.

$$P_r = P_r(A_{ij}, B_{ij}, D_{ij}, k_1, k_2) \quad (r = 0, \ldots, 4)$$

The equation $\Delta_1(\gamma^2) = 0$ has the two purely imaginary roots $\gamma_{1,2} = \pm ik_1$, corresponding to the generating solution (5-22). Detaching these two roots, we get a polynomial of degree six

$$\Delta_1^*(\gamma^2) = -P_o^*\gamma^6 + P_1^*\gamma^4 - P_2^*\gamma^2 + P_3^* = 0 \qquad (5\text{-}26)$$

such that

$$\Delta_1(\gamma^2) = \Delta_1^*(\gamma^2)(\gamma^2 + k_1^2)$$

where

$$P_s^* = P_s^*(A_{ij}, B_{ij}, D_{ij}, k_1, k_2), \quad (s = 0, \ldots, 3)$$

Eq. (5-26) is the basic equation which determines completely the properties of the dynamic edge effect for free vibration having the wave numbers k_1 and k_2. All roots of the equation $\Delta_1^*(\gamma^2) = 0$ are to be either real or complex conjugate which can be denoted by $\gamma = \pm\, \alpha_k + i\beta_k$ $(k = 2, 3, 4)$, where α_k and β_k are functions of the wave numbers k_1 and k_2. We discard the roots having positive real parts, since the corresponding solutions do not contribute significantly to the natural mode near the edge $x = 0$ (refer to Chapter IV). Thus the corrective solution, which is called the dynamic edge effect, for the natural mode near the edge $x = 0$ can be written and classified as follows [16].

(i) If all β_k are zero and all α_k $(k = 2, 3, 4)$ are distinct positive real, then the corrective solution can be written in the form

$$\Psi_3(x|0) = C_3 e^{-\alpha_2 x} + C_4 e^{-\alpha_3 x} + C_5 e^{-\alpha_4 x} \tag{5-27}$$

Since all terms decay (as x increases) and do not oscillate, we call this dynamic edge effect non-oscillatory.

(ii) For multiple real positive roots, i.e., $\alpha_4 = \alpha_3 \neq \alpha_2$, all $\alpha_k > 0$, and all β_k are zero, then

$$\Psi_3(x|0) = C_3 e^{-\alpha_2 x} + C_4 e^{-\alpha_3 x} + C_5 x\, e^{-\alpha_3 x} \tag{5-28}$$

The dynamic edge effect remains non-oscillatory and decays as x increases.

(iii) Assume $\alpha_2 > 0$, $\beta_2 = 0$, $\alpha_4 = \alpha_3 > 0$, and $\beta_3 = -\beta_4 > 0$, then

$$\Psi_3(x|0) = C_3 e^{-\alpha_2 x} + C_4 e^{-\alpha_3 x} \sin \beta_3 x + C_5 e^{-\alpha_3 x} \cos \beta_3 x \quad (5\text{-}29)$$

which is called an oscillatory dynamic edge effect with amplitude decaying as x increases.

All three types (i), (ii), and (iii) of dynamic edge effect are such that the influence of the boundary conditions on the form of the mode shape decays as the distance from the boundary increases; therefore, the solution for the natural mode approaches the generating solution toward the internal region.

(iv) If there cannot be found three roots with negative real parts in the equation $\Delta_1^*(\gamma^2) = 0$, the dynamic edge effect will be called degenerate. This means that the influence of the boundary conditions on the mode shape will not decay with increasing the distance from the boundary; therefore, this influence is considerable everywhere even in the internal region. In this case, the asymptotic method is invalid.

Now, we let $\xi = -\gamma^2$, then Eq. (5-26) becomes

$$\Delta_1^*(-\xi) = P_o^* \xi^3 + P_1^* \xi^2 + P_2^* \xi + P_3^* = 0 \quad (5\text{-}30)$$

In order that among the roots of the equation $\Delta_1^*(\gamma^2) = 0$ there will be three roots with negative real parts, it is necessary and sufficient that the equation $\Delta_1^*(-\xi) = 0$ has no positive real or zero roots. This statement can be proved as follows.

(i) Sufficiency: If the equation $\Delta_1^*(-\xi) = 0$ has no positive real or zero roots, the roots must be in the form

$$-\xi_1,\ -\xi_2,\ -\xi_3 \quad \text{or} \quad -\xi_1,\ \alpha \pm i\beta$$

where ξ_1, ξ_2, ξ_3 are positive real and α, β are real. Since $\gamma^2 = -\xi$, hence the roots of the equation $\Delta_1^*(\gamma^2) = 0$ will be

$$\pm\sqrt{\xi_1}\ ,\quad \pm\sqrt{\xi_2}\ ,\quad \pm\sqrt{\xi_3}$$

$$\text{or}\quad \pm\sqrt{\xi_1}\ ,\quad \pm(\alpha^2+\beta^2)^{1/4}\left[\sqrt{1+\alpha/\sqrt{\alpha^2+\beta^2}} \pm i\sqrt{1-\alpha/\sqrt{\alpha^2+\beta^2}}\right]$$

which contain three roots with negative real parts.

(ii) <u>Necessity</u>: If the equation $\Delta_1^*(-\xi) = 0$ has at least one positive real root, i.e., its roots will be in the form

$$\xi_1,\ -\xi_2,\ -\xi_3 \quad \text{or} \quad \xi_1,\ \alpha \pm i\beta$$

where ξ_1 is positive real and ξ_2, ξ_3, α, β are real. Then the roots of the equation $\Delta_1^*(\gamma^2) = 0$ will be

$$\pm i\sqrt{\xi_1}\ ,\quad \pm\sqrt{\xi_2}\ ,\quad \pm\sqrt{\xi_3}$$

$$\text{or}\quad \pm i\sqrt{\xi_1}\ ,\quad \pm(\alpha^2+\beta^2)^{1/4}\left[\sqrt{1+\alpha/\sqrt{\alpha^2+\beta^2}} \pm i\sqrt{1-\alpha/\sqrt{\alpha^2+\beta^2}}\right]$$

If the equation $\Delta_1^*(-\xi) = 0$ has one zero root, say $\xi_1 = 0$, then the equation $\Delta_1^*(\gamma^2) = 0$ has double zero roots. For either case, the equation $\Delta_1^*(\gamma^2) = 0$ can not have three (but at most two) roots with negative real parts.

The dynamic edge effect has not been found to degenerate for any of the laminated plate systems investigated by the writer. For cross-ply plates consisting of similar orthotropic layers simple conditions on material are given in Appendix C.

For a nondegenerate dynamic edge effect the asymptotic solution, which is the sum of the generating solution and the corrective solution, for the natural mode in the x direction near the edge x = 0 is given by

$$\Phi_3(x|0) = C_1 \sin k_1 x + C_2 \cos k_1 x + \Psi_3(x|0) \tag{5-31}$$

and with similar expressions for $\Phi_1(x|0)$ and $\Phi_2(x|0)$. Here $\Psi_i(x|0)$ $(i = 1,2,3)$ are the corrective solutions expressed in the form as (5-27) or (5-28) or (5-29) according to the property of the dynamic edge effect. Each function $\Phi_i(x|0)$ involves five constants. From the governing equations, the coefficients associated in $\Phi_1(x|0)$ and $\Phi_2(x|0)$ can be related with the coefficients in $\Phi_3\ (x|0)$. Consequently, there are five unknown constants C_k $(k = 1,\ldots,5)$ in the displacement field. The ratios between these five constants can be solved as functions of k_1 and k_2 from the four boundary conditions at $x = 0$. Therefore, the asymptotic solution for natural mode near the edge $x = 0$ is obtained once k_1 and k_2 are found.

Similarly, the asymptotic solution for natural mode in the x direction near the boundary x = a can be expressed in the form

$$\Phi_3(x|a) = C_1^* \sin k_1(a-x) + C_2^* \cos k_1 (a-x) + \Psi_3(x|a) \tag{5-32}$$

with the similar expressions for $\Phi_1(x|a)$ and $\Phi_2(x|a)$. The corrective solution $\psi_i(x|a)$ is written in the same form as $\psi_i(x|o)$ with x replaced by a - x. Now, the ratios between the five unknown constants C_k^* $(k = 1,\ldots,5)$ are determined from the four boundary conditions at $x = a$.

With the same procedure, the asymptotic solutions for natural mode in the y direction near the boundaries $y=0$ and $y=b$ can be obtained. To find the asymptotic solutions for natural mode in the y direction, we only need to exchange the corresponding terms between the x- and y-directions (i.e. exchange X, U, a, 1 and y, V, b, 2) if the formulation for the solution in the x coordinate has been set. The asymptotic solution of natural mode for the entire region, except in the neighborhood of corners, of the plate is obtained by joining these solutions arising from four boundaries. All these asymptotic solutions arising from four boundaries will tend to the generating solution as the distance from each boundary increases.

By matching the two solutions for natural mode near the edges $x=0$ and $x=a$ in the internal region with corrective solutions neglected, we can find the following relation

$$C_1 \sin k_1 x + C_2 \cos k_1 x = C_1^* \sin k_1 (a-x) + C_2^* \cos k_1 (a-x) \qquad (5\text{-}33)$$

Thus, a formula for the wave number k_1 is obtained

$$k_1 a = \arctan\left(-\frac{C_2}{C_1}\right) + \arctan\left(-\frac{C_2^*}{C_1^*}\right) + m\pi, \quad (m=0,1,\ldots) \qquad (5\text{-}34)$$

Similarly, a formula for the wave number k_2 is given by

$$k_2 a = \arctan\left(-\frac{D_2}{D_1}\right) + \arctan\left(-\frac{D_2^*}{D_1^*}\right) + n\pi, \quad (n=0,1,\ldots) \qquad (5\text{-}35)$$

where ratios D_2/D_1 and D_2^*/D_1^* are to be determined from the boundary conditions at $y=0$ and $y=b$, respectively. Since all ratios C_2/C_1, C_2^*/C_1^*,

D_2/D_1, and D_2^*/D_1 are functions of k_1 and k_2 (unknown parameters), the wave numbers k_1 and k_2 can be solved from the two transcendental equations (5-34) and (5-35).

As before, let $\mu = k_1/k_2$ ($k_2 \neq 0$) and $R = b/a$, then a single equation for the wave number ratio μ can be obtained in the form

$$\mu = R\,\frac{\arctan S_{11}(\mu) + \arctan S_{12}(\mu) + m\pi}{\arctan S_{21}(\mu) + \arctan S_{22}(\mu) + n\pi}\,,\ (m,n = 0,1\ldots) \qquad (5\text{-}36)$$

where

$$S_{11}(\mu) = -C_2/C_1,\ S_{12}(\mu) = -C_2^*/C_1^*,\ S_{21}(\mu) = -D_2/D_1,\ \text{and}$$
$$S_{22}(\mu) = -D_2^*/D_1^*\ .$$

From Eq. (5-23), we can find

$$\lambda/k_2^4 = D_{11}\mu^4 + 2(D_{12} + 2D_{66})\mu^2 + D_{22} - \frac{1}{J_1^*}[(A_{66}\mu^2 + A_{22})J_2^{*2}\mu^2 \qquad (5\text{-}37)$$
$$- 2(A_{12} + A_{66})J_2^* J_3^* \mu^2 + (A_{11}\mu^2 + A_{66})J_3^{*2}]$$

where

$$J_1^* = (A_{11}\mu^2 + A_{66})(A_{66}\mu^2 + A_{22}) - (A_{12} + A_{66})^2\mu^2 \qquad (5\text{-}38)$$

$$J_2^* = B_{11}\mu^2 + (B_{12} + 2B_{66})$$

$$J_3^* = (B_{12} + 2B_{66})\mu^2 + B_{22}$$

Let $\eta = \gamma/k_2$, then Eq. (5-26) becomes

$$\Box_1(\eta^2) = Q_0\eta^8 + Q_1\eta^6 + Q_2\eta^4 + Q_3\eta^2 + Q_4 = 0 \qquad (5\text{-}39)$$

where

$$Q_r = Q_r\ (A_{ij},\ B_{ij},\ D_{ij},\ \mu) = P_r/k_2^{2r},\quad (r = 0,\ldots,4)$$

and the equation $\Delta_1^*(\gamma^2) = 0$ becomes

$$\Box_1^*(\eta^2) = -Q_o^*\eta^6 + Q_1^\mu\eta^4 - Q_2^*\eta^2 + Q_3^* = 0 \qquad (5\text{-}40)$$

such that

$$\Box_1(\eta^2) = \Box_1^*(\eta^2)(\eta^2 + \mu^2)$$

where

$$Q_s^* = Q_s^*(A_{ij},\ B_{ij},\ D_{ij},\ \mu),\ (s = 0,\ldots,3)$$

Therefore, ratios C_2/C_1 and C_2^*/C_1^* will be functions of μ (the only unknown parameter). Similarly, D_2/D_1 and D_2^*/D_1^* are also functions of μ. Once μ is solved from Eq. (5-36), the wave numbers can be obtained from the following two equations

$$k_1a = \arctan S_{11}(\mu) + \arctan S_{12}(\mu) + m\pi,\ (m=0,1,\ldots) \quad (5\text{-}41)$$

$$k_2b = \arctan S_{21}(\mu) + \arctan S_{22}(\mu) + n\pi,\ (n=0,1,\ldots) \quad (5\text{-}42)$$

The approximate value for a natural frequency thus can be obtained from Eq. (5-37), or (5-23).

The procedure of numerical computation for a given plate (i.e., material properties and plate dimensions are given) with a specific set of boundary conditions at each edge is as follows. Using a successive iteration method, Eq. (5-36) can be considered as $\mu_{i+1} = f(\mu_i)$.

Starting from a proper initial value of μ (e.g., $\mu_o = \frac{m+0.5}{n+0.5}$) for any mode having $s_x = m + 1$ and $s_y = n + 1$ nodal lines running "perpendicular" to the x- and y-directions, respectively, we can solve Eq. (5-40) for the corrective solution by noting that $\gamma_k = k_2\eta_k$. The sum of the generating solution and the corrective solution must satisfy the four conditions at $x = 0$ or at $x = a$ to give four homogeneous equations for five unknown constants C_k or C_k^* ($k = 1,\ldots,5$). Taking out the common factor (k_2^α, α is certain integer) in each equation, the ratios C_k/C_1 or C_k^*/C_1^* (here $k = 2,\ldots,5$) can be determined, i.e., $-C_2/C_1 = S_{11}(\mu_o),\ldots$ etc. Similarly, D_2/D_1 and D_2^*/D_1^* can be found as $-D_2/D_1 = S_{21}(\mu_o)$ and $-D_2^*/D_1^* = S_{22}(\mu_o)$. Substituting these values into the right hand side of Eq. (5-36), we get a new value $\mu_1 = f(\mu_o)$ for the next iteration. Repeated iterations yield rapid convergence to a certain value of μ. Once μ is obtained to a desired accuracy, the wave numbers k_1 and k_2 can be obtained immediately. Subsequently, an approximate solution for natural frequency and an asymptotic solution for natural mode can be obtained.

Numerical solutions for natural frequencies and modal stresses are obtained for cross-ply rectangular plates consisting of even number of identical orthotropic layers with plies oriented at 0° and 90° to the x-axis alternately. Comparison between asymptotic solution and exact solution for natural frequency is shown in Table 7 for three cases of two-layer cross-ply plates with edges $y = 0$ and $y = b$ simply-supported S2. In Table 7, comparison between two solutions for modal stresses is also shown for the case of a two-layer square plate with edges $x = 0$ and $x = a$ rigidly-clamped. The errors obtained by the asymptotic solution for frequency compared with the exact solutions are very small for all modes except the lowest mode having one nodal line running "parallel" to the two longer free edges. For example, the error of asymptotic solution for the natural frequency of the lowest mode having one nodal line running "parallel" to the free edges $x = 0$ and $x = a$ of a two-layer cross-ply plate is about 1.5% when the aspect ratio $b/a = 1$, but this error is about 20% when $b/a = 2$. This is because the dynamic edge effect decays (in the x coordinate) slowly (the predominant term in the corrective solution is $C_3 e^{-\alpha_2 x} = -0.68 C e^{-1.85x/a}$) and the contribution of k_1 to the frequency is important for the latter case (i.e., $b/a = 2$). The errors obtained by asymptotic solution for bending moments along the rigidly-clamped edge are at the same order as those of the corresponding frequencies.

Fundamental frequency as a function of the aspect ratio of a cross-ply plate (material properties $E_\ell/E_t = 40$, $G_{\ell t}/E_t = 0.5$, $\nu_{\ell t} = 0.25$) is illustrated in Figure 9 for two cases of boundary conditions. The size of coupling effect as a function of E_ℓ/E_t on fundamental frequency of cross-ply square plate is shown in Figure 10 for certain cases of boundary conditions. It is apparent that the coupling effect essentially depends on the ratio E_ℓ/E_t and the total number of layers in the plate.

Antisymmetric Angle-Ply Laminated Plates

The antisymmetric angle-ply rectangular plates under investigation satisfy the basic assumptions in Chapter II, hence the governing equation of free transverse vibration are given by (4-5) and the boundary conditions can be any combination of Eqs. (2-29) and (2-30). Eq. (4-5) will be satisfied by assuming

$$\begin{bmatrix} U \\ V \\ W \end{bmatrix} = \begin{bmatrix} E \sin k_1 (x - x_o) \cos k_2(y - y_o) \\ F \cos k_1 (x - x_o) \sin k_2(y - y_o) \\ G \sin k_1 (x - x_o) \sin k_2(y - y_o) \end{bmatrix} \tag{5-43}$$

Substituting (5-43) into (4-5), a non-trivial solution yields an algebraic equation for the natural frequency

$$\begin{aligned} \lambda = {} & D_{11}k_1^4 + 2(D_{12} + 2D_{66})k_1^2k_2^2 + D_{22}k_2^4 - \frac{1}{J_4}\,[(A_{11}k_1^2 \\ & + A_{66}k_2^2)J_5^2k_1^2 - 2(A_{12} + A_{66})J_5J_6k_1^2k_2^2 + \\ & (A_{66}k_1^2 + A_{22}k_2^2)\,J_6^2k_2^2] \end{aligned} \tag{5-44}$$

where J_4, J_5, and J_6 are defined as the same forms in (4-18) and (4-21).

With the same procedure as that for unsymmetric cross-ply plates discussed in preceding section, we can obtain the asymptotic solutions for natural modes and estimates of natural frequencies for antisymmetric angle-ply plates.

Numerical results for natural frequencies and modal stresses are obtained for angle-ply plates consisting of even number of identical orthotropic layers. Comparison between asymptotic solution and exact solution for natural frequency is shown in Table 9 for two types of two-layer angle-ply square plates with edges y = 0,b simply-supported S3 and three cases of boundary conditions at the other two edges. The agreement between asymptotic solutions and exact solutions are excellent for all modes except the modes which behave similar to cylindrical bending. The error associated with the asymptotic solution is less for higher modes since the dynamic edge effect decays faster for higher modes than for the lower modes. Comparison between the asymptotic solution and the exact solution for modal stresses is shown in Table 8 for a two-layer angle-ply square plate with edges $y = 0$, b simply-supported S3 and edges $x = 0$, a rigidly-clamped. The errors associated with the asymptotic solutions for bending moments along the edges are very small and are of the same order as those of the corresponding frequencies.

Natural frequencies of different modes as functions of the aspect ratio for a two-layer $75^\circ/-75^\circ$ angle-ply plate with edges $x = 0$, a free and edges $y = 0$, b rigidly-clamped are illustrated in Figure 11. Natural frequency as a function of the ply-orientation θ are shown in Figure 12 for a mode having two nodal lines running "parallel" to both x- and y-axes of a two-layer antisymmetric angle-ply square plate for certain cases of boundary conditions.

The size of coupling effect as a function of the ply-orientation θ on the fundamental frequency of an antisymmetric angle-ply square plate is illustrated in Figure 13 for various boundary conditions. It appears that the coupling effect essentially depends on the ply-orientation and the total number of layers in the plate and is essentially independent of the boundary conditions.

The effect of inplane boundary conditions on the laminate response is investigated for certain types of angle-ply square plates having all edges clamped. Fundamental frequencies for certain types of angle-ply plates ($E_\ell/E_t = 40$, $G_{\ell t}/E_t = 0.5$, $\nu_{\ell t} = 0.25$) are listed in Table 12 for three different clamped boundary conditions C1, C2, and C3. The effect of inplane boundary conditions is not significant for antisymmetric angle-ply plates with plies oriented at + θ and -θ alternately. But for a laminate with plies oriented at 45°/-30°/30°/-45°, the fundamental frequency of C2 plate is reduced by 20% compared with the value for a C1 plate. Similar to the results obtained in Chapter IV by closed form solution, the fundamental natural frequency of a rigidly-clamped (C1) plate is higher (or at least not less) than those of C2 and C3 plates. This is contrary to the result as shown in Table 12 obtained by Whitney [8] that the rigidly-clamped condition reduces the stiffness of angle-ply plates compared to the less rigid clamp. Since the set of admissible functions (also the comparison functions) for natural modes of a rigidly-clamped plate is a subset of the set of admissible functions for a less rigidly-clamped plate. Minimizing the Rayleigh quotient with respect to the coefficients of admissible functions, the minimum value of the Rayleigh quotient for the subset is not less than the value for the wider set. This analytical conclusion agrees with the present results for the effect of inplane boundary conditions.

A comparison between asymptotic solution and a Ritz solution for natural frequencies of a four-layer (45°/45°/-45°/-45°) glass-fiber plate with all edges rigidly-clamped, Case 1, is shown in Table 13. Exact solution for natural frequencies of the same plate with two opposite edges simply-supported S3 and the other two edges rigidly-clamped, Case 2, is also added for comparison. It is surprising that the Ritz solutions obtained by Bert and Mayberry [6] are not only generally lower than the experimental results, in which eight values they got for four types of laminated plates are all higher than their Ritz solutions except the fundamental mode of the case shown in Table 13, but also lower than the values obtained by the asymptotic solution for Case 1 and the values by the exact solution for Case 2. As mentioned in [6], the Ritz method should give an upper bound; also transverse shear deformation, interlaminar shear deformation, and rotary inertias (all of which would reduce the frequency) have all been neglected in the analytical solutions. Furthermore, the natural frequency for Case 1 should be higher than that for Case 2.

CHAPTER VI

CONCLUSIONS

Boundary value problems of unsymmetric laminated rectangular plates based on the classical Kirchhoff assumptions of small deformation theory have been formulated in terms of displacements. The resulting governing equations revealed a coupling phenomenon between transverse deflection and inplane displacements in the middle plane of the plate which is not found in the linear theory of homogeneous plates.

The significance of inplane inertia effects on natural frequencies of both simply-supported cross-ply and angle-ply laminated plates has been investigated. Results showed that inplane inertia effects on the predominantly transverse vibration are not significant for the lower modes of laminated thin plates. The main effect of inplane inertias is therefore to make the analysis of the inplane resonance phenomenon possible.

A convenient method for finding closed form solutions for natural modes, frequencies, and modal stresses in free transverse vibrations of laminated rectangular thin plates having a pair of simply-supported opposite edges was presented. This method is similar to the Levy method, but is comparatively simpler in formulation and provides a fast technique of computation for natural frequencies.

A simple method based on the concept of dynamic edge effects for finding asymptotic solutions for natural modes, frequencies, and modal stresses of rectangular plates has been discussed. Results indicated that this method gives satisfactory solutions for all modes except for the modes whose half-wave lengths in both directions are simultaneously longer than the shorter side of the plate. The faster the edge effects decay, the less is the error of the asymptotic solution. The dynamic edge effects decay more rapidly with increasing the wave numbers. As a result, in contrast to the other approximate methods, the asymptotic solution is more accurate for higher modes. Furthermore, this method greatly reduces the computation work compared with other methods.

Various results substantiate the conclusions of Whitney [5, 7, 8] that the effect of bending-stretching coupling decreases natural frequencies compared to the analogous decoupled plate, and that this coupling effect depends on the degree of anisotropy (i.e., the ratio E_{ℓ}/E_t), the ply-orientation of individual layers, and the total number of layers in the laminated plates. The coupling effect reduces the bending moments while increasing the stress resultants as compared with the analogous decoupled plate. Results also showed that the effect of boundary conditions is a minor factor on the bending-stretching coupling on dynamic laminate response.

For certain arrangements of antisymmetric angle-ply laminated plates (e.g. a four-layer plate with four identical layers oriented at $45°/-30°/30°/-45°$), the effect of inplane boundary conditions is significant on laminate response (the differences between natural frequencies corresponding to different inplane boundary conditions are large). As a result, the application of the "reduced bending stiffness method" should be discouraged, since the inplane boundary conditions are disregarded in this method. However, the laminate response for cross-ply plates could be well approximated by the "reduced bending stiffness method."

It was found, for plates having a pair of free opposite edges, that the cylindrical bending solution for the lowest natural frequency of a given half-wavelength paralleling to the free edges will produce a great error when the ply-orientation angle of individual layers in the antisymmetric angle-ply plate becomes large with the free edges. For cross-ply plates or for antisymmetric angle-ply plates with small ply-orientation angle to the free edges, the error of cylindrical bending solution is very small.

It has been shown, e.g. [29, 30], that the classical linear theory is adequate for studying the flexural response of laminated plate in every respect for long wavelengths (compared to the plate thickness). However, for a plate with short wavelengths (compared to the plate thickness) the use of classical linear theory is deficient and a higher order theory for more accurate solution must be employed. Some theories for the dynamic response of laminated plates for this purpose can be found from the literature recently presented by Sun and Whitney [30]. Perhaps a useful extension of this work would be an assessment of the utility of the asymptotic method for such a higher order theory.

Table 1. Frequencies of Simply-Supported Laminated Plates with and without Inplane Inertia Effects

$(E_\ell/E_t = 40,\ G_{\ell t} = 0.5,\ \nu_{\ell t} = 0.25)$

Cross-Ply Plate

$\frac{\ell_x}{h} = \frac{\ell_y}{h}$	L	ω^*	Ω_1^*	Ω_2^*	Ω_3^*
10	2	11.164	11.100	142.3	146.6
10	4	17.216	17.191	141.7	146.6
10	12	18.635	18.632	141.5	146.6
30	2	11.164	11.156	424.7	439.9
50	2	11.164	11.162	707.6	733.1
50	4	17.216	17.215	707.5	733.1
50	12	18.635	18.635	707.4	733.1

Antisymmetric 45° Angle-ply Plate

$\frac{\ell_x}{h} = \frac{\ell_y}{h}$	L	ω^*	Ω_1^*	Ω_2^*	Ω_3^*
10	2	14.637	14.556	31.42	202.5
30	2	14.637	14.629	94.25	604.3
50	2	14.637	14.634	157.1	1006.8
50	4	23.528	23.527	157.1	1006.6
50	12	25.575	25.575	157.1	1006.6

L indicates the total number of layers in the plate,

$(\omega^*,\ \Omega_i^*) = (\omega,\ \Omega_i)\ell_x^{\,2}(\rho/E_t h^3)^{1/2},\quad (i = 1,\ 2,\ 3)$

Table 2. Amplitude Ratios Associated with Different Vibration Modes of Simply-Supported Laminated Plates Including Inplane Inertias

$(E_\ell/E_t = 40,\ G_{\ell t}/E_t = 0.5,\ \nu_{\ell t} = 0.25)$

$\frac{\ell_x}{h}, \frac{\ell_y}{h}$	L	1st Mode A^*	1st Mode B^*	2nd Mode A^*	2nd Mode B^*	3rd Mode A^*	3rd Mode B^*
Cross-ply Plate							
25, 50	2	0.0300	-0.0156	-1.5	61.1	-33.8	-0.8
25, 50	4	0.0150	-0.0078	-3.0	121.7	-67.5	-1.7
50, 50	2	0.0151	-0.0151	-33.0	33.0	∞	∞
50, 50	4	0.0076	-0.0076	-65.8	65.8	∞	∞
50, 50	12	0.0025	-0.0025	-194.3	194.3	∞	∞
100, 50	2	0.0078	-0.0150	-122.2	3.0	1.7	67.5
100,100	2	0.0076	-0.0076	-65.9	65.9	∞	∞
5° Angle-Ply Plate							
50, 50	2	0.0031	0.0284	1.2	-35.3	-251.6	-8.8
100,100	2	0.0015	0.0141	2.4	-71.0	-505.0	-17.7
45° Angle-Ply Plate							
25, 50	2	0.0143	0.0306	58.1	-59.7	-22.7	-22.1
25, 50	4	0.0071	0.0153	115.4	-118.7	-45.3	-44.1
50, 50	2	0.0149	0.0149	∞	$-\infty$	-33.5	-33.5
50, 50	4	0.0075	0.0075	∞	$-\infty$	-66.9	-66.9
50, 50	12	0.0025	0.0025	∞	$-\infty$	-200.8	-200.8
100, 50	2	0.0153	0.0071	-119.7	116.5	-44.1	-45.4
100,100	2	0.0075	0.0075	$-\infty$	∞	-66.9	-66.9

L indicates the total number of layers in the plate.

$A^* = A/C$, $B^* = B/C$.

Table 3. Fundamental Frequencies ---$\omega b^2(\rho/E_t h^3)^{1/2}$---of Laminated Square Plates with $y = 0$, b Simply-Supported and $x = 0$, a Clamped in Various Inplane Boundary Conditions

($E_\ell/E_t = 40$, $G_{\ell t}/E_t = 0.5$, $\nu_{\ell t} = 0.25$)

	Two-layer Cross-ply Plate			Four-layer Cross-ply Plate		
	S2-C1	S2-C2	S2-C3	S2-C1	S2-C2	S2-C3
	18.724	18.723	18.723	29.563	29.563	29.563
	S3-C1	S3-C2	S3-C3	S3-C1	S3-C2	S3-C3
	Two-layer θ/-θ Plate			Four-layer θ/-θ/θ/-θ Plate		
5°	36.683	36.682	36.248	40.122	40.122	40.023
25°	23.230	23.229	23.137	35.800	35.780	35.785
45°	19.395	19.394	19.395	30.774	30.773	30.774
55°	17.692	17.617	17.661	27.373	27.360	27.368
65°	16.312	15.825	16.300	23.832	23.750	23.829
75°	16.177	15.326	16.176	20.953	20.788	20.953
85°	18.777	17.978	18.776	19.787	19.601	19.787
	Four-layer 45°/-60°/60°/-45°			Six-layer 85°/85°/-5°/5°/-85°/-85°		
	24.647	24.000	23.045	20.913	19.778	20.828

Table 4. Comparison of Exact Solution and Cylindrical Bending Solution for Frequencies of Homogeneous Orthotropic Plates with $y = 0$, b Simply-Supported and $x = 0$, a Free.

		$D_x = D_y = H$ (Isotropic)			
	ℓ/a, b/a \ ν	0.1	0.2	0.3	0.4
ω_B^*		9.820	9.670	9.415	9.046
ω_E^*	4.0	9.824	9.685	9.444	9.091
	2.0	9.832	9.717	9.511	9.203
	1.0	9.846	9.771	9.631	9.408
	0.5	9.857	9.817	9.735	9.599
ω_C^*		9.870			
		$D_x/H = 1.0$, $D_y/H = 2.0$			
	ℓ/a, b/a \ ν, ν_{yx}	0.1	0.2	0.3	0.4
ω_B^*		13.923	13.817	13.640	13.388
ω_E^*	4.0	13.926	13.828	13.660	13.419
	2.0	13.932	13.850	13.707	13.494
	1.0	13.941	13.888	13.790	13.635
	0.5	13.949	13.920	13.863	13.767
ω_C^*		13.958			

$(\omega^* = \omega b^2 \sqrt{\rho/H})$

Table 5. The Characteristic Roots γ_k and Their Coefficients for a Two-Layer Cross-Ply Square Plate with $y = 0$, b Simply-Supported S2

$(E_\ell/E_t = 40,\ G_{\ell t} = 0.5,\ \nu_{\ell t} = 0.25)$

Case 1. Rigidly-Clamped at $x = 0$ and $x = a$						
Mode No.	(1,1)	(2,1)	(1,2)	(2,2)	(1,3)	(2,3)
$\kappa_1 a$	4.6567	7.8277	4.4725	7.7565	4.2549	7.6537
$\alpha_2 a$	4.9918	8.0317	5.7390	8.5496	6.8706	9.3636
$\alpha_3 a$	0.4906	0.4906	0.9808	0.9811	1.4697	1.4714
$\alpha_4 a$	20.115	20.115	40.230	40.230	60.345	60.345
C_3/C	0.6825	0.6966	0.6150	0.6720	0.5261	0.6328
C_4/C	2×10^{-5}	4×10^{-7}	3×10^{-4}	1×10^{-5}	1×10^{-3}	10×10^{-5}
C_5/C	-3×10^{-5}	-7×10^{-5}	-7×10^{-6}	-2×10^{-5}	-3×10^{-6}	-8×10^{-6}

Case 2. Free at $x = 0$ and $x = a$						
Mode No.	(2,1)	(3,1)	(2,2)	(3,2)	(2,3)	(3,3)
$\kappa_1 a$	2.1358	4.8933	2.6041	5.2169	2.8605	5.5120
$\alpha_2 a$	2.7919	5.2133	4.4403	6.3363	6.1068	7.7124
$\alpha_3 a$	0.4904	0.4906	0.9795	0.9809	1.4667	1.4707
$\alpha_4 a$	20.115	20.115	40.230	40.230	60.345	60.345
C_3/C	-0.5694	-0.6794	-0.3716	-0.6003	-0.2539	-0.4990
C_4/C	7×10^{-5}	4×10^{-8}	2×10^{-3}	5×10^{-6}	2×10^{-3}	5×10^{-5}
C_5/C	-7×10^{-7}	1×10^{-6}	9×10^{-6}	3×10^{-7}	-2×10^{-6}	-9×10^{-9}

$C = (C_1^2 + C_2^2)^{1/2}$

Table 6. The Characteristic Roots γ_k and Their Coefficients for a Two-Layer 45°/-45° Square Plate with y = 0, b Simply-Supported S3

$(E_\ell/E_t = 40,\ G_{\ell t}/E_t = 0.5,\ \nu_{\ell t} = 0.25)$

Case 1. Rigidly-Clamped at x = 0 and x = a

Mode No.	(1,1)	(2,1)	(1,2)	(2,2)	(1,3)	(2,3)
$k_1 a$	4.0807	7.5528	3.6389	7.1417	3.4676	6.8929
$\alpha_2 a$	6.0475	10.2544	14.3412	15.6026	21.0946	21.9198
$\alpha_3 a$	0.9578	0.9578	1.9160	1.9157	2.8740	2.8741
$\beta_3 a$	2.9922	2.9920	5.9841	5.9844	8.9760	8.9763
C_3/C	0.4523	0.5930	0.2460	0.4162	0.1622	0.3000
C_4/C	-4×10^{-5}	-7×10^{-6}	1×10^{-4}	-9×10^{-5}	7×10^{-5}	2×10^{-4}
C_5/C	8×10^{-5}	2×10^{-5}	1×10^{-4}	3×10^{-5}	6×10^{-5}	1×10^{-4}

Case 2. Free at x = 0 and x = a

Mode No.	(2,1)	(3,1)	(2,2)	(3,2)	(2,3)	(3,3)
$k_1 a$	2.6486	5.4453	2.6695	5.7677	2.5459	5.8334
$\alpha_2 a$	7.4245	8.8181	14.1263	15.0234	20.9628	21.6100
$\alpha_3 a$	0.9580	0.9578	1.9160	1.9160	2.8739	2.8741
$\beta_3 a$	2.9922	2.9921	5.9840	5.9844	8.9760	8.9761
C_3/C	-0.2906	-0.4790	-0.2073	-0.3075	-0.1846	-0.2422
C_4/C	-2×10^{-4}	-7×10^{-5}	-7×10^{-7}	-3×10^{-4}	1×10^{-6}	-2×10^{-5}
C_5/C	-3×10^{-4}	5×10^{-5}	-4×10^{-5}	-3×10^{-4}	-1×10^{-5}	-1×10^{-4}

$C = (C_1^2 + C_2^2)^{1/2}$

Table 7. Comparison of Asymptotic Solution and Exact Solution for Frequencies and Modal Stresses of Cross-Ply Plates with y = 0, b Simply-Supported S2

$(E_\ell/E_t = 40,\ G_{\ell t}/E_t = 0.5,\ \nu_{\ell t} = 0.25)$

		Two-Layer Square Plate with x = 0, a Rigidly-Clamped				
Mode No.		(1,1)	(1,2)	(2,1)	(2,2)	(1,3)
ω^*	Asymp	18.635	34.886	47.192	57.225	69.327
	Exact	18.724	34.913	47.185	57.221	69.334
$-N^*_{xo}$	Asymp	0.078	0.535	0.051	0.358	1.585
	Exact	0.347	1.303	0.010	0.137	2.785
$-N^*_{xc}$	Asymp	0.046	0.474	0.044	0.248	1.505
	Exact	0.325	1.338	0.000	0.001	3.044
$-M^*_{xo}$	Asymp	17.357	17.711	48.261	49.144	18.498
	Exact	17.521	17.651	48.248	49.182	18.265
M^*_{xc}	Asymp	11.159	10.716	-0.434	-0.314	10.459
	Exact	10.531	10.334	0.000	0.000	10.411

Frequencies ---ω^*--- of Two-Layer Cross-ply plate with x = 0, a Free

b/a = 1.0			b/a = 2.0		
Mode No.	Asymp	Exact	Mode No.	Asymp	Exact
(1,1)	7.319	7.311	(1,1)	7.319	7.310
(2,1)	8.675	8.550	(2,1)	13.472	11.504
(3,1)	20.212	20.281	(1,2)	29.275	29.243
(1,2)	29.275	29.249	(2,2)	34.700	34.200
(2,2)	30.559	30.509	(1,3)	65.869	65.804
(3,2)	38.164	38.177	(3,1)	68.768	69.196

$\omega^* = \omega b^2(\rho/E_t h^3)^{1/2}$

Table 8. Comparison of Asymptotic Solution and Exact Solution for Modal Stresses of Angle-ply Square Plates with y = 0,b Simply-Supported S3 and x = 0, a Rigidly-Clamped

$(E_\ell/E_t = 40,\ G_{\ell t}/E_t = 0.5,\ \nu_{\ell t} = 0.25)$

Two-Layer 45°/-45° Plate						
Mode No.		(1,1)	(1,2)	(2,1)	(1,3)	(2,2)
N^*_{xo}	Asymp	0.508	0.065	0.139	-0.120	1.972
	Exact	0.501	0.240	0.094	-0.227	2.028
N^*_{xc}	Asymp	-0.003	-4.545	-0.043	1.800	1.737
	Exact	-0.193	-2.298	-0.077	-4.142	3.765
$-M^*_{xo}$	Asymp	11.756	17.447	30.740	23.974	39.250
	Exact	11.794	17.504	30.689	23.956	39.286
M^*_{xc}	Asymp	7.863	14.572	0.063	26.747	-0.224
	Exact	7.826	16.049	0.023	26.338	0.456

Two-Layer 65°/-65° Plate						
Mode No.		(1,1)	(2,1)	(1,2)	(3,1)	(2,2)
N^*_{xo}	Asymp	4.784	2.603	7.780	-0.017	17.169
	Exact	4.346	3.449	7.764	-0.581	17.207
N^*_{xc}	Asymp	2.146	-4.547	-0.220	2.266	-20.112
	Exact	5.952	-7.147	-2.307	9.590	-13.326
$-M^*_{xo}$	Asymp	4.603	11.106	8.550	21.581	16.662
	Exact	4.573	11.135	8.552	21.570	16.652
M^*_{xc}	Asymp	3.713	-0.589	7.862	-14.402	-0.818
	Exact	2.887	0.377	7.770	-15.249	0.093

Table 9. Comparison of Asymptotic Solution and Exact Solution for Frequencies of Angle-ply Square Plates with $x = 0$, a Simply-Supported S3

Frequency $\omega b^2(\rho/E_t h^3)^{1/2}$, $(E_\ell/E_t = 40,\ G_{\ell t}/E_t = 0.5,\ \nu_{\ell t} = 0.25)$

Two-Layer 45°/-45° Plate

I	Mode No.	(1,1)	(1,2)	(2,1)	(1,3)	(2,2)
	Asymp	19.391	37.011	44.158	65.082	66.867
	Exact	19.395	37.011	44.158	65.081	66.867
II	Mode No.	(1,1)	(2,1)	(1,2)	(3,1)	(2,2)
	Asymp	5.582	12.446	22.328	27.725	30.877
	Exact	4.980	12.445	21.039	27.726	30.878
III	Mode No.	(1,1)	(2,1)	(1,2)	(2,2)	(3,1)
	Asymp	8,397	23.883	24.586	45.621	47.764
	Exact	8.382	23.883	24.591	45.617	47.764

Two-Layer 65°/-65° Plate

I	Mode No.	(1,1)	(2,1)	(1,2)	(3,1)	(2,2)
	Asymp	16.312	31.570	42.198	53.160	60.146
	Exact	16.311	31.619	42.198	53.151	60.147
II	Mode No.	(1,1)	(2,1)	(3,1)	(1,2)	(4,1)
	Asymp	8.650	12.914	22.431	34.598	37.381
	Exact	8.270	12.907	22.430	33.914	37.357
III	Mode No.	(1,1)	(2,1)	(3,1)	(1,2)	(2,2)
	Asymp	10.111	19.653	34.562	35.669	47.946
	Exact	10.423	19.684	34.559	36.323	48.497

Case I: Rigidly-Clamped at $x = 0$ and $x = a$.
Case II: Free at $x = 0$ and $x = a$.
Case III: Rigidly-Clamped at $x = 0$ and Free at $x = a$.

Table 10. Comparison of Asymptotic Solution and Other Solutions for Frequencies of Homogeneous Orthotropic Square Plates

(Frequency $\omega b^2 \sqrt{\rho/H}$)

		C-C-S-S		C-C-C-C			
Mode No.		(1,1)		(1,1)		(3,1)	
$\frac{D_x}{H}$	$\frac{D_y}{H}$	Asymp	Ref.[23]	Asymp	Ref.[24]	Asymp	Ref.[24]
0.5	0.5	21.978	22.493	27.473	28.071	98.665	98.612
	1.0	24.551	25.194	31.559	32.271	100.099	100.411
	2.0	28.994	29.794	38.537	39.293	102.643	103.127
1.0	1.0	26.867	27.10	35.092	35.965	131.629	131.581
	2.0	30.968	31.910	41.416	42.397	133.560	133.965
2.0	2.0	34.576	35.681	46.832	47.958	180.145	180.145

$D_x/H = 1.543$, $D_y/H = 4.810$, $D_{xy}/H = 0.407$ (Five-ply maple)							$D_x = D_y = H$, $\nu = 0.3$ (isotropic)	
C-C-C-C			C-F-C-F					
Mode No.	Asymp.	Ref.[24]	Mode No.	Asymp	Ref.[24]	Asymp	Ref.[24]	
(1,1)	57.865	58.980	(1,1)	27.584	27.730	22.207	22.148	
(2,1)	95,988	96.916	(1,2)	32,794	31.634	26.732	26.346	
(1,2)	141.647	142.009	(1,3)	66.188	64.558	44.563	43.405	
(3,1)	164.767	165.353	(2,1)	76.623	76.404	61.685	61.058	
(2,2)	167.534	168.341	(2,2)	82.501	81.774	67.287	66.957	

Table 11. Comparison of Asymptotic Solution and Other Solutions for Frequencies of Homogeneous Orthotropic Cantilever Plates

Clamped edge: $y=0$. Frequency $\omega b^2\sqrt{\rho/D_x}$

$D_y/D_x = 3.12$, $H/D_x = 0.648$, $D_{xy}/D_x = 0.2637$ (Five-ply Maple)					
$a/b = 0.5$			$a/b = 1.0$		
Mode No.	Asymptotic	Ref.[25]	Mode No.	Asymptotic	Ref.[25]
(1,1)	4.358	11.769	(1,1)	4.358	6.209
(2,1)	16.791	14.996	(2,1)	8.241	9.532
(1,2)	39.225	38.877	(3,1)	26.594	27.024
(2,2)	53.639	55.273	(1,2)	39.225	38.924
(3,1)	92.653	93.304	(2,2)	43.240	

$D_y/D_x = 1.265$, $H/D_x = 0.96$, $D_{xy}/D_x = 0.339$ (Steel)					
$a/b = 0.5$			$a/b = 1.0$		
Mode No.	Asymptotic	Ref.[25]	Mode No.	Asymptotic	Ref.[25]
(1,1)	2.775	3.921	(1,1)	2.775	3.938
(2,1)	16.628	14.999	(2,1)	7,886	8.714
(1,2)	24.976	24.451	(1,2)	24.976	24.042
(2,2)	47.389	49.767	(3,1)	26.392	27.699
(1,3)	69.378	68.451	(2,2)	32.011	33.268

$D_x = D_y = H$, $\nu = 0.3$ (Isotropic) $a/b = 1.0$					
Mode No.	Asymptotic	Upper Bound		Lower Bound	
(1,1)	2.467	3.473	[28]	3.430	[28]
(2,1)	7.788	8.547	[26]	7.260	[27]
(1,2)	22.207	21.304	[28]	20.874	[28]
(3,1)	26.268	27.291	[28]	26.501	[28]
(2,2)	30.064	31.17	[26]	28.546	[27]

Table 12. Comparison of Fundamental Frequencies of Angle-Ply Square Clamped Plates with Various Inplane Boundary Conditions

$(E_\ell/E_t = 40,\ G_{\ell t}/E_t = 0.5,\ \nu_{\ell t} = 0.25)$

Frequency $\omega b^2(\rho/E_t h^3)^{1/2}$

θ	C1	C2	C3
	Two-Layer Plate, θ/-θ		
5°	36.880 (37.582)	36.051	36.401 (39.458)
15°	27.378 (28.052)	26.825	27.156 (31.337)
25°	24.317 (24.940)	24.003	24.215 (27.517)
35°	23.189 (23.799)	23.007	23.111 (25.316)
45°	22.838 (23.488)	22.830	22.835 (23.696)
	Four-Layer Plate, θ/-θ/θ/-θ		
5°	40.267	40.098	40.156
15°	38.044	37.946	38.006
25°	37.025	36.974	37.010
35°	36.351	36.324	36.340
45°	36.086	36.085	36.086
	Four-Layer Plate, 45°/-30°/30°/-45°		
	33.077	26.779	28.425
	Six-Layer Plate, 85°/85°/-5°/5°/-85°/-85°		
	40.600	39.296	39.401

Values in parentheses are obtained by Whitney [8].

Table 13. Comparison of Asymptotic Solution and Ritz Solution for Frequencies of Glass-Fiber Angle-Ply Plates

$(45^\circ/45^\circ/-45^\circ/-45^\circ)$, $a = 9$ in, $b = 6$ in

Frequency in Hertz

	Case 1: C1-C1-C1-C1			Case 2: S3-C1-S3-C1
Mode No.	Asymptotic	Ritz Sol. [6]	Experiment [6]	Exact Sol.
(1,1)	205	180	167	194
(2,1)	325	280		280
(1,2)	504	425		495
(3,1)	517	430		434
(2,2)	626	512		597
(4,1)	778			657
(3,2)	822	660		762
(1,3)	949	795	860	942
(2,3)	1076	875		1053
(4,2)	1086			991
(5,1)	1105			947
(3,3)	1279	1005		1231

Properties of a single layer:

$E_\ell = 2.695 \times 10^6$ psi

$E_t = 2.56 \times 10^6$ psi

$\nu_{\ell t} = 0.242$

$G_{\ell t} = 0.6 \times 10^6$ psi

$\rho_o = 0.000197$ lb-sec^2/in^4

$h_s = 0.0105$ in

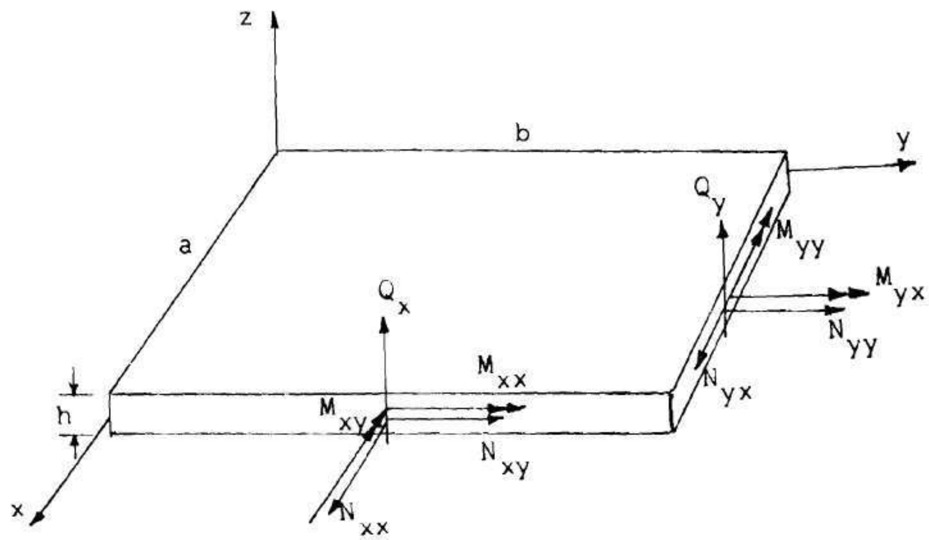

Figure 1. Geometry, Coordinate Systems, and Sign Conventions of the Plates.

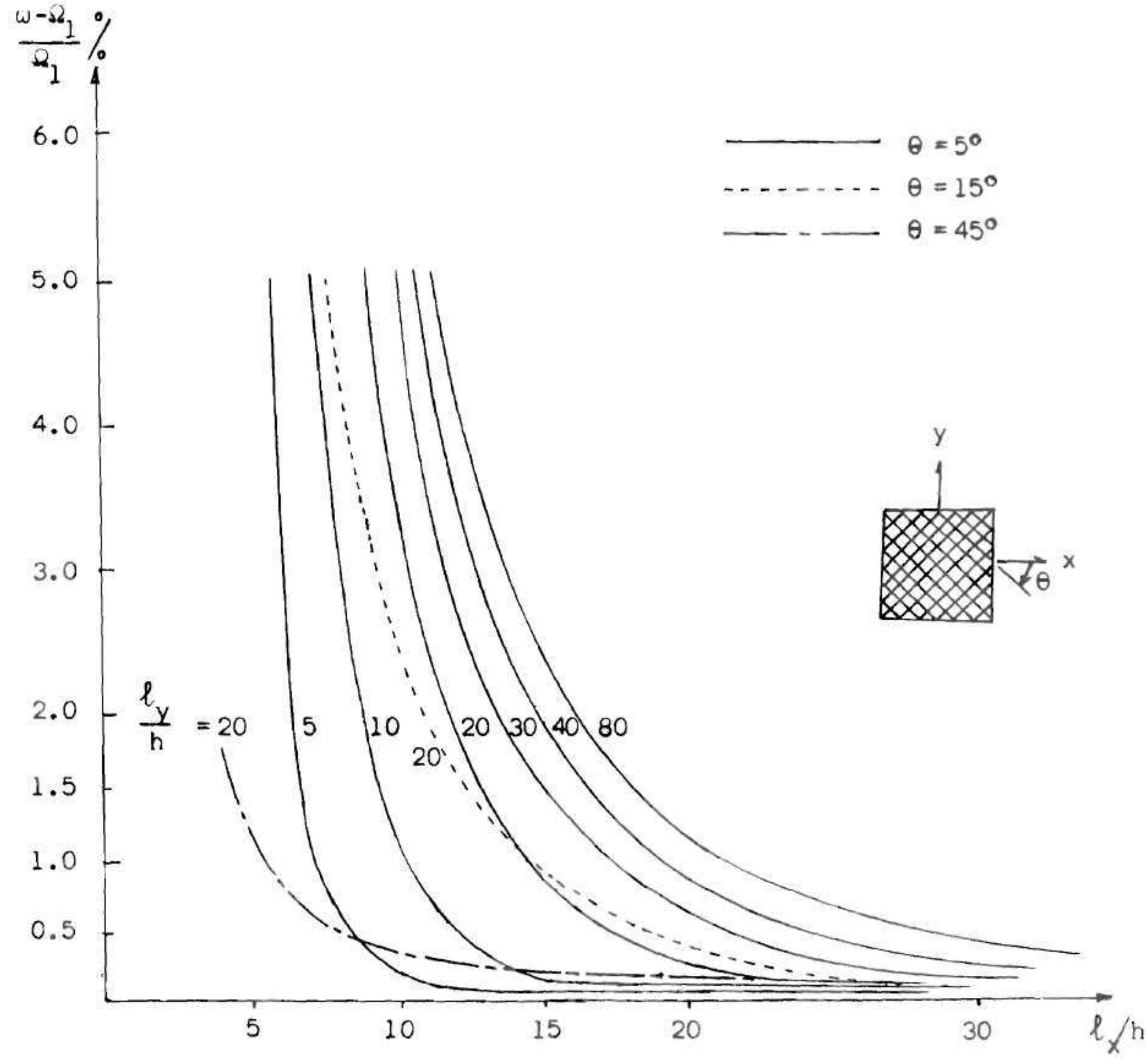

Figure 2. Error of ω Compared with Ω_1 for Two-Layer Simply-Supported Angle-Ply Plates.

(E_ℓ/E_t = 40, $G_{\ell t}/E_t$ = 0.5, $\nu_{\ell t}$ = 0.25)

D. S. = Decoupled Solution
C. S. = Coupled Solution

——— Case 1: Cl at x = 0 and x = a
—-—- Case 2: F at x = 0 and x = a

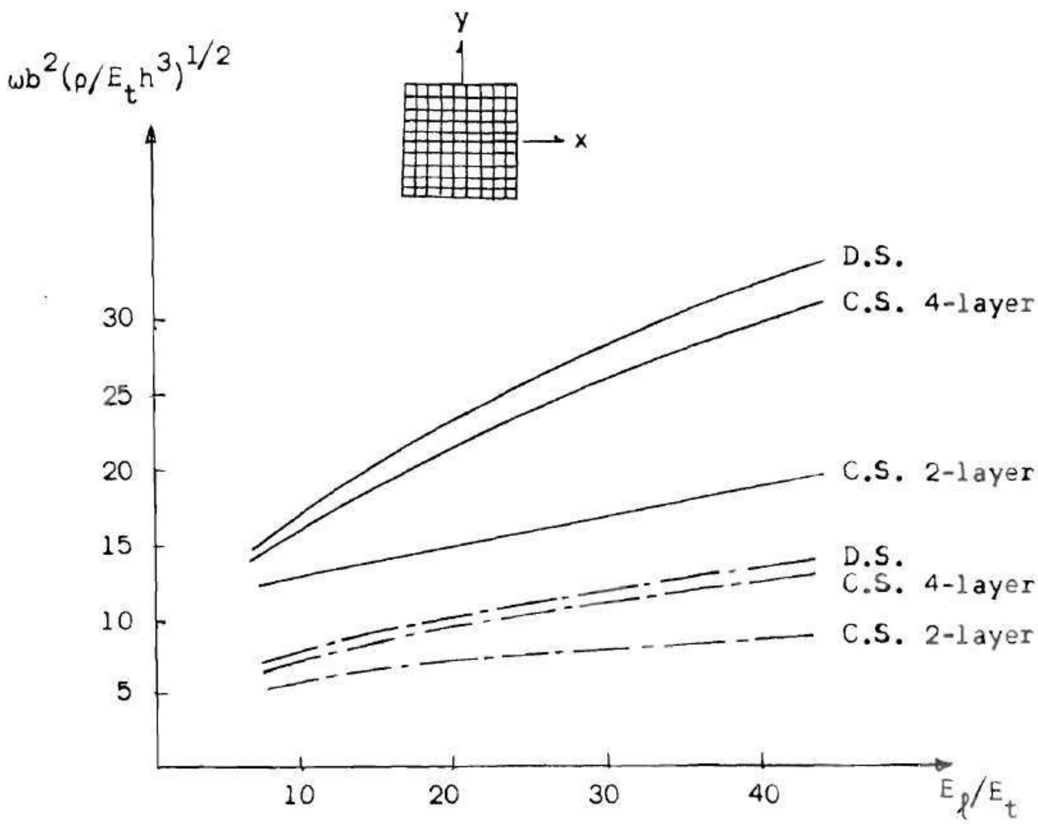

Figure 3. Frequency of Mode $s_x = s_y = 2$ as a Function of E_ℓ/E_t for a Cross-Ply Square Plate with $y = 0$, b Simply-Supported S2.

$(G_{\ell t}/E_t = 0.5, \quad \nu_{\ell t} = 0.25)$

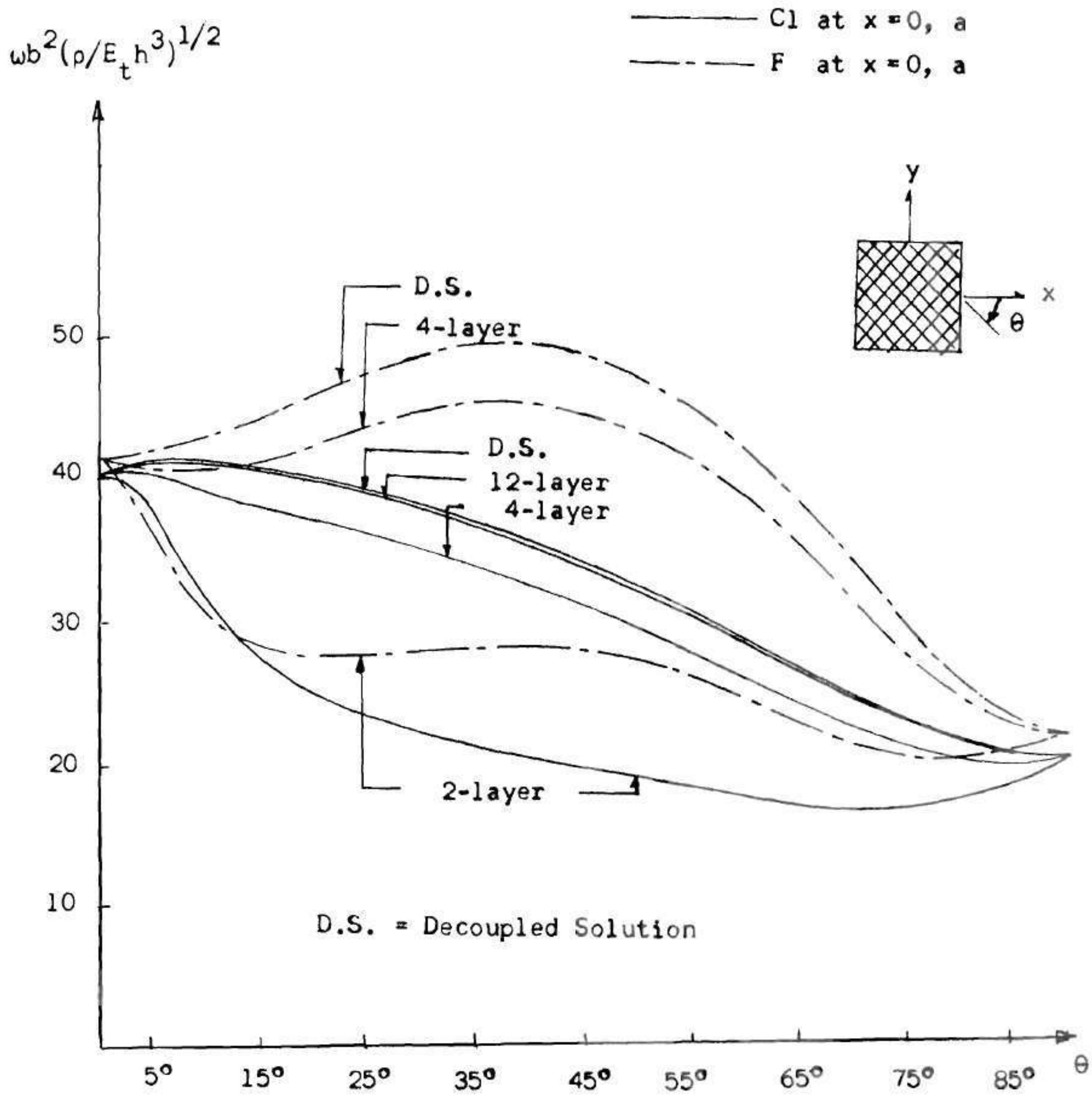

Figure 4. Frequency of Mode $s_x = s_y = 2$ as a Function of θ for an Angle-Ply Square Plate with y = 0, b Simply-Supported S3.

($E_\ell/E_t = 40$, $G_{\ell t}/E_t = 0.5$, $\nu_{\ell t} = 0.25$)

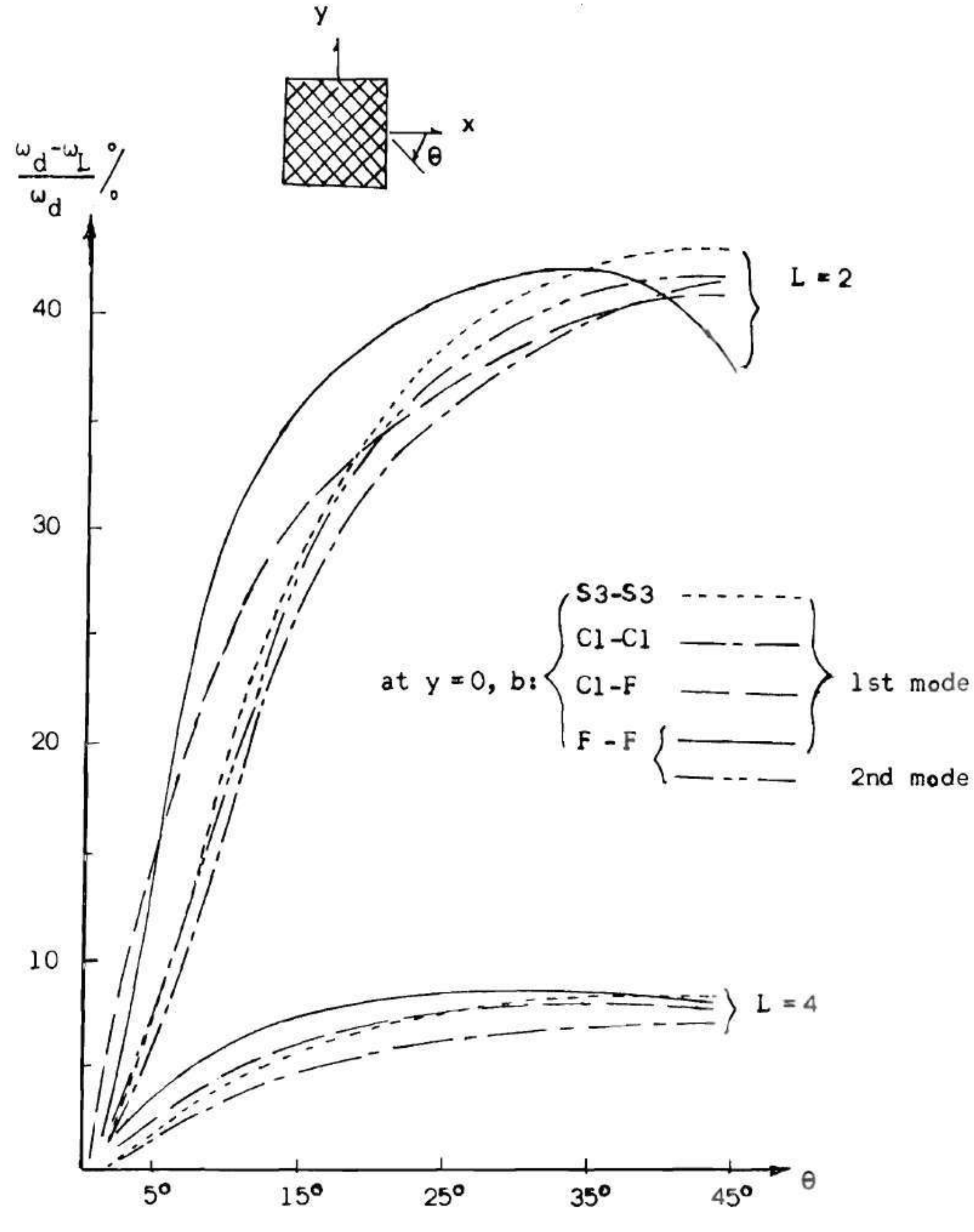

Figure 5. Coupling Effect on Frequencies of Angle-Ply Square Plates with x = 0, a Simply-Supported S3.

($E_\ell/E_t = 40$, $G_{\ell t}/E_t = 0.5$, $\nu_{\ell t} = 0.25$)

L indicates the total number of layers in the plate.

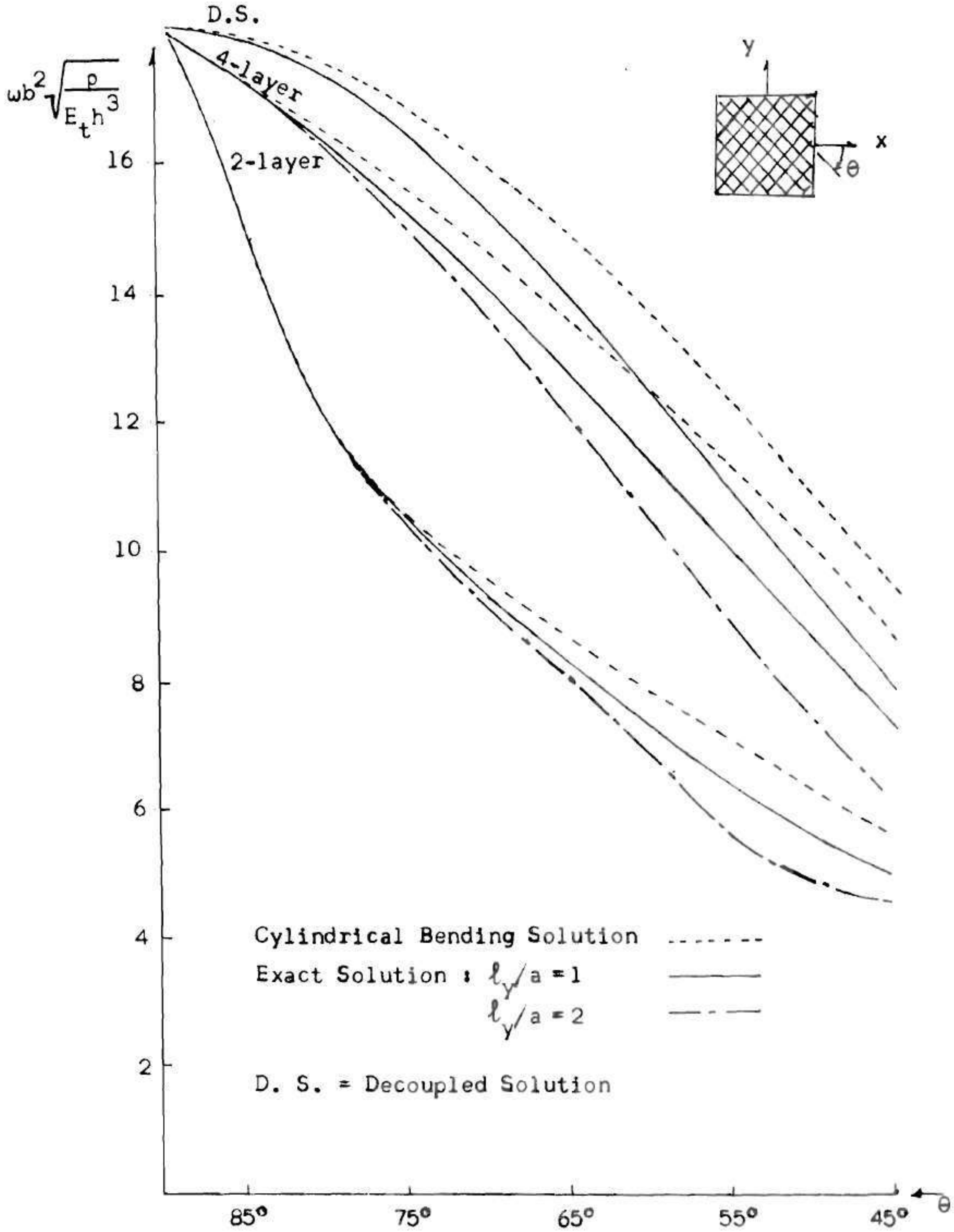

Figure 6. Comparison of Exact Solution and Cylindrical Bending Solution for Frequencies of Angle-Ply Plates with y = 0, b Simply-Supported S3 and x = 0, a Free.

$(E_\ell/E_t = 40, \;\; G_{\ell t}/E_t = 0.5, \;\; \nu_{\ell t} = 0.25)$

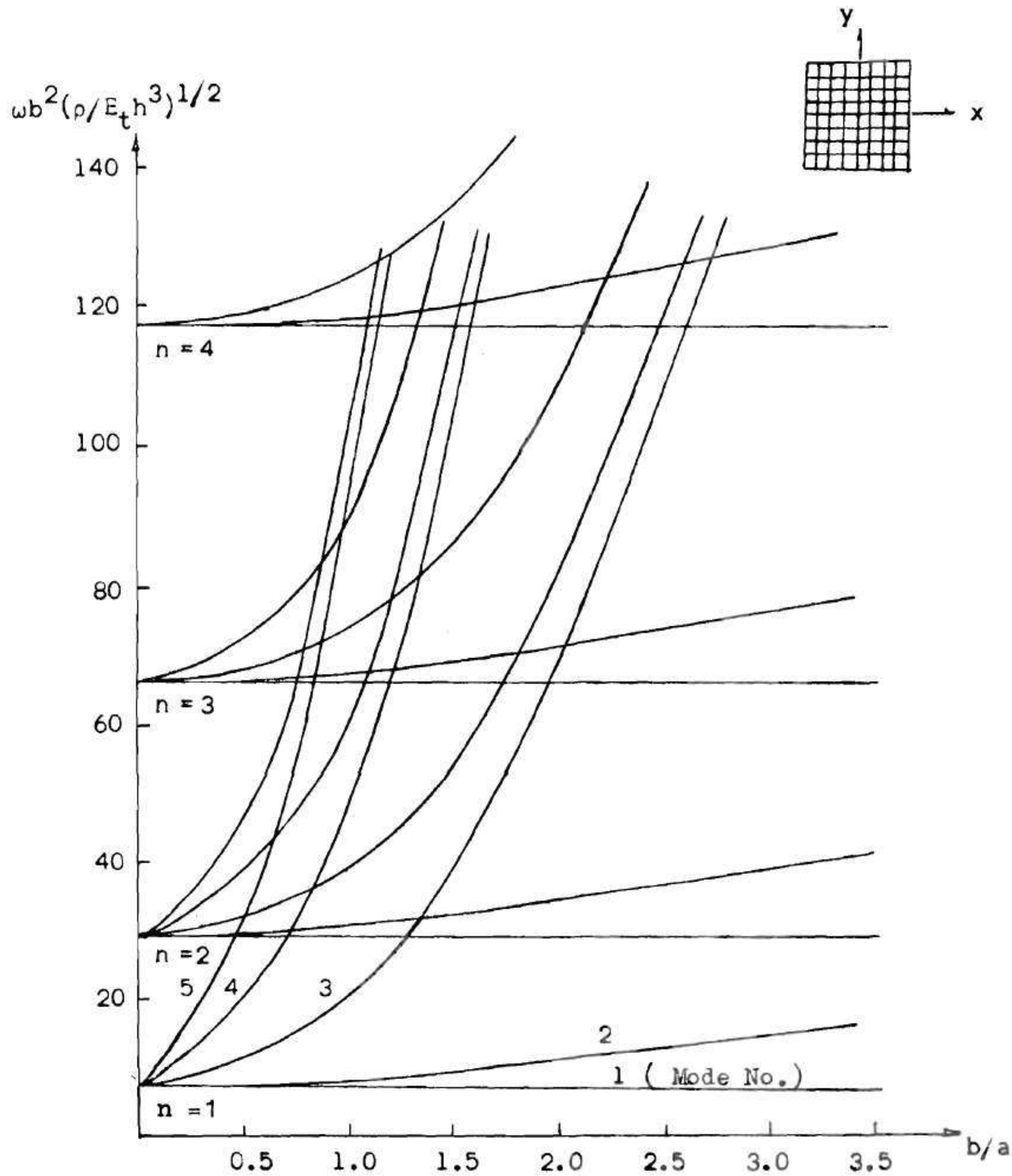

Figure 7. Frequencies of a Two-Layer Cross-Ply Plate with $y = 0$, b Simply-Supported S2 and $x = 0$, a Free.

$(E_\ell / E_t = 40, \quad G_{\ell t}/E_t = 0.5, \quad \nu_{\ell t} = 0.25)$

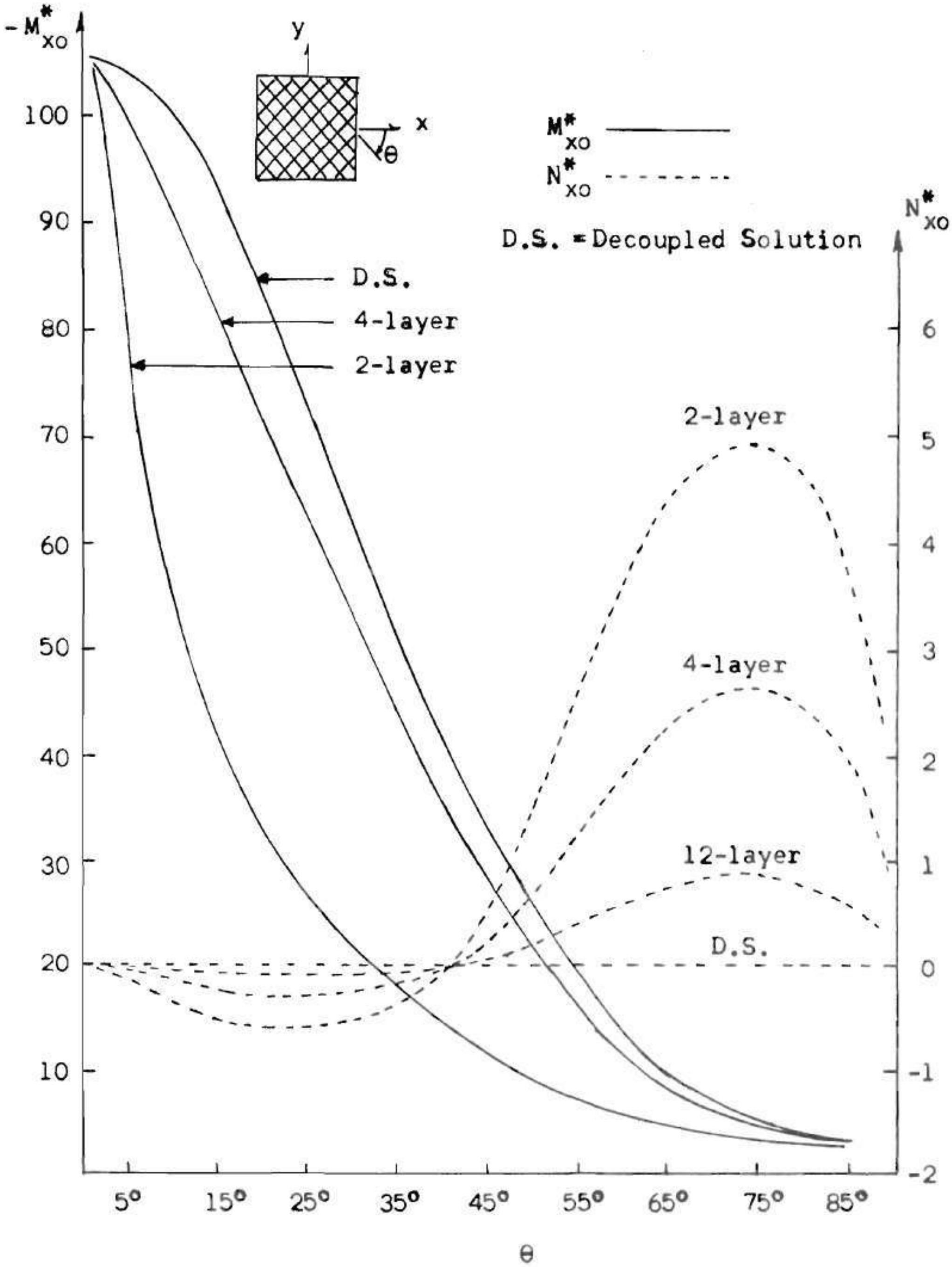

Figure 8. Modal Stresses for the Fundamental Modes of Angle-Ply Square Plates with y = 0, b Simply-Supported S3 and x = 0, a Rigidly-Clamped.

(E_ℓ/E_t = 40, $G_{\ell t}/E_t$ = 0.5, $\nu_{\ell t}$ = 0.25)

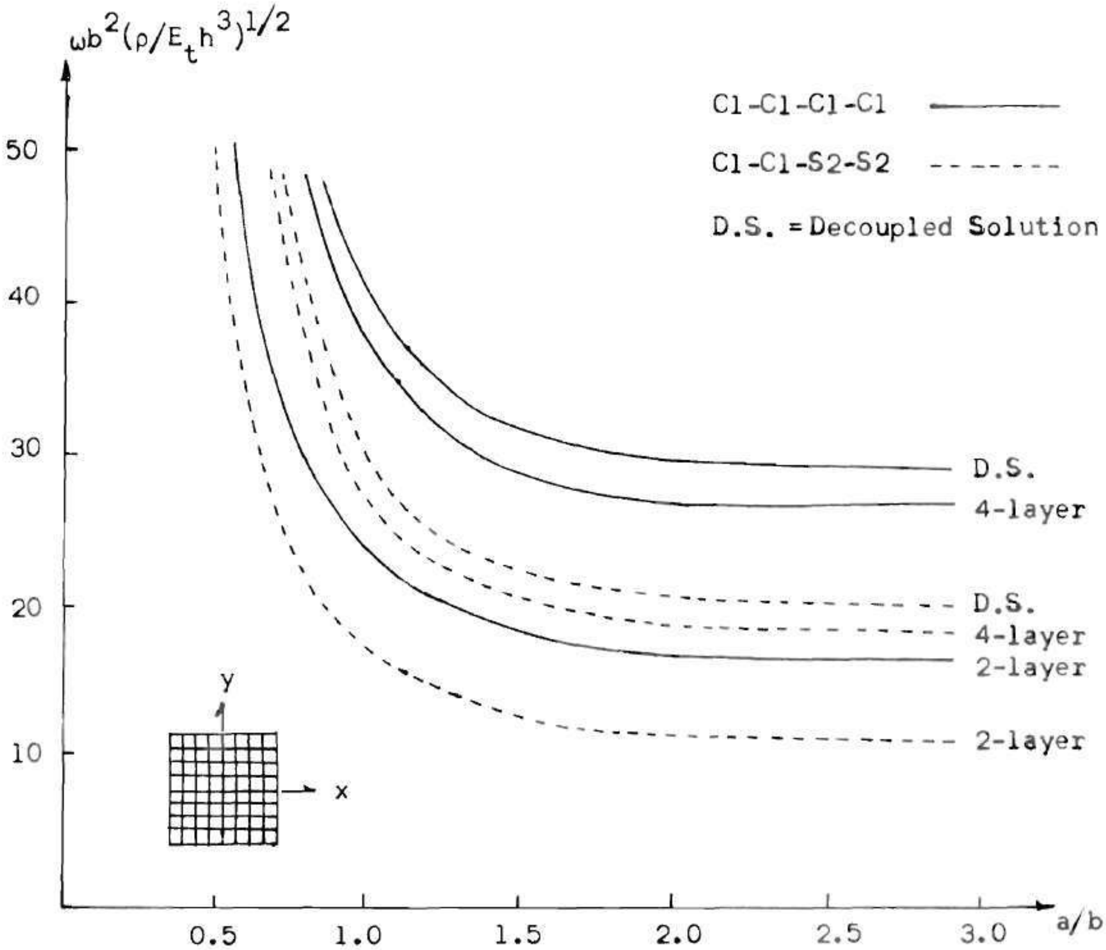

Figure 9. Fundamental Frequency as a Function of Aspect Ratio for a Cross-Ply Plate.

$(E_\ell/E_t = 40, \quad G_{\ell t}/E_t = 0.5, \quad \nu_{\ell t} = 0.25)$

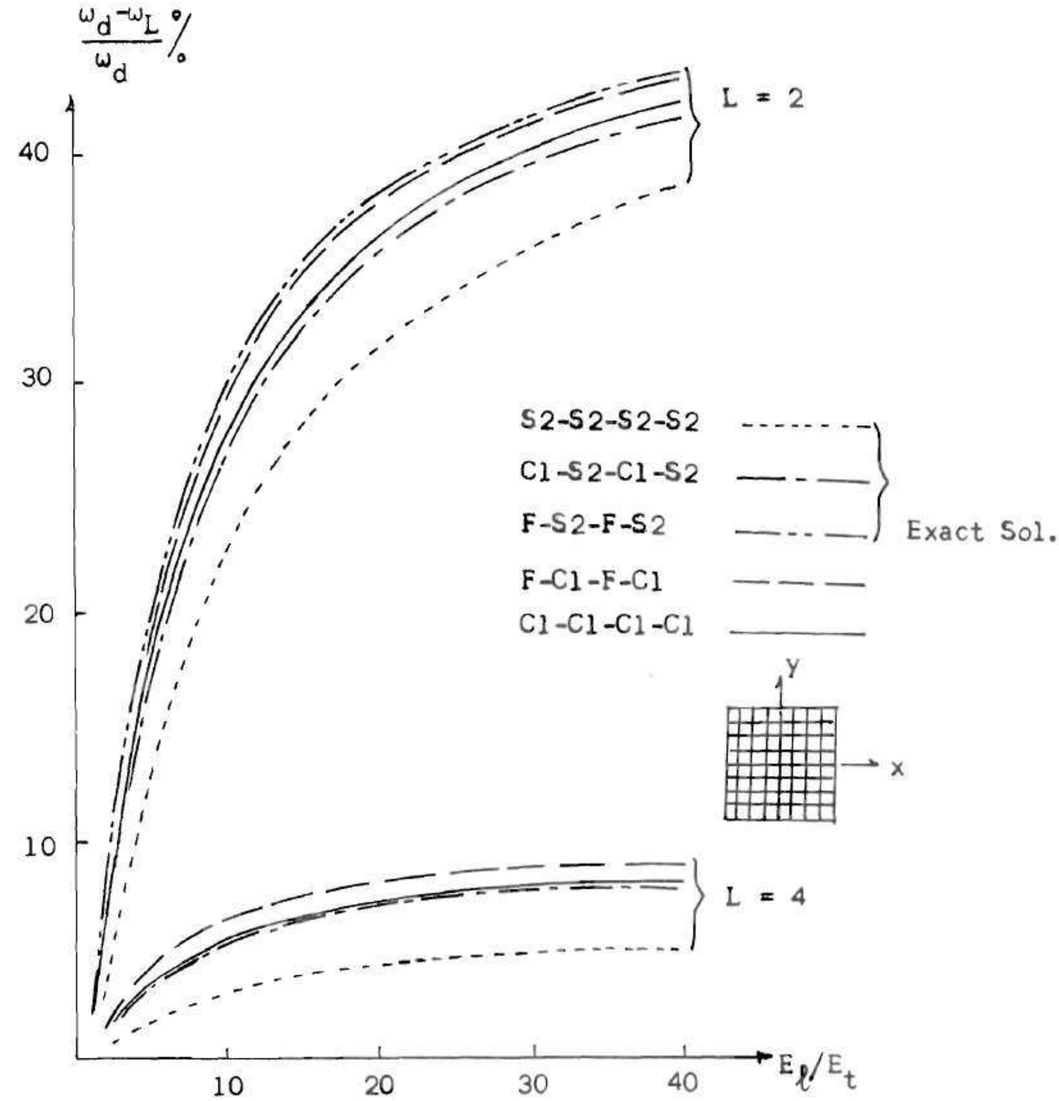

Figure 10. Coupling Effects on Fundamental Frequencies of Cross-Ply Square Plates with Various Boundary Conditions.

($G_{\ell t}/E_t = 0.5$, $\nu_{\ell t} = 0.25$)

L indicates the total number of layers in the plate.

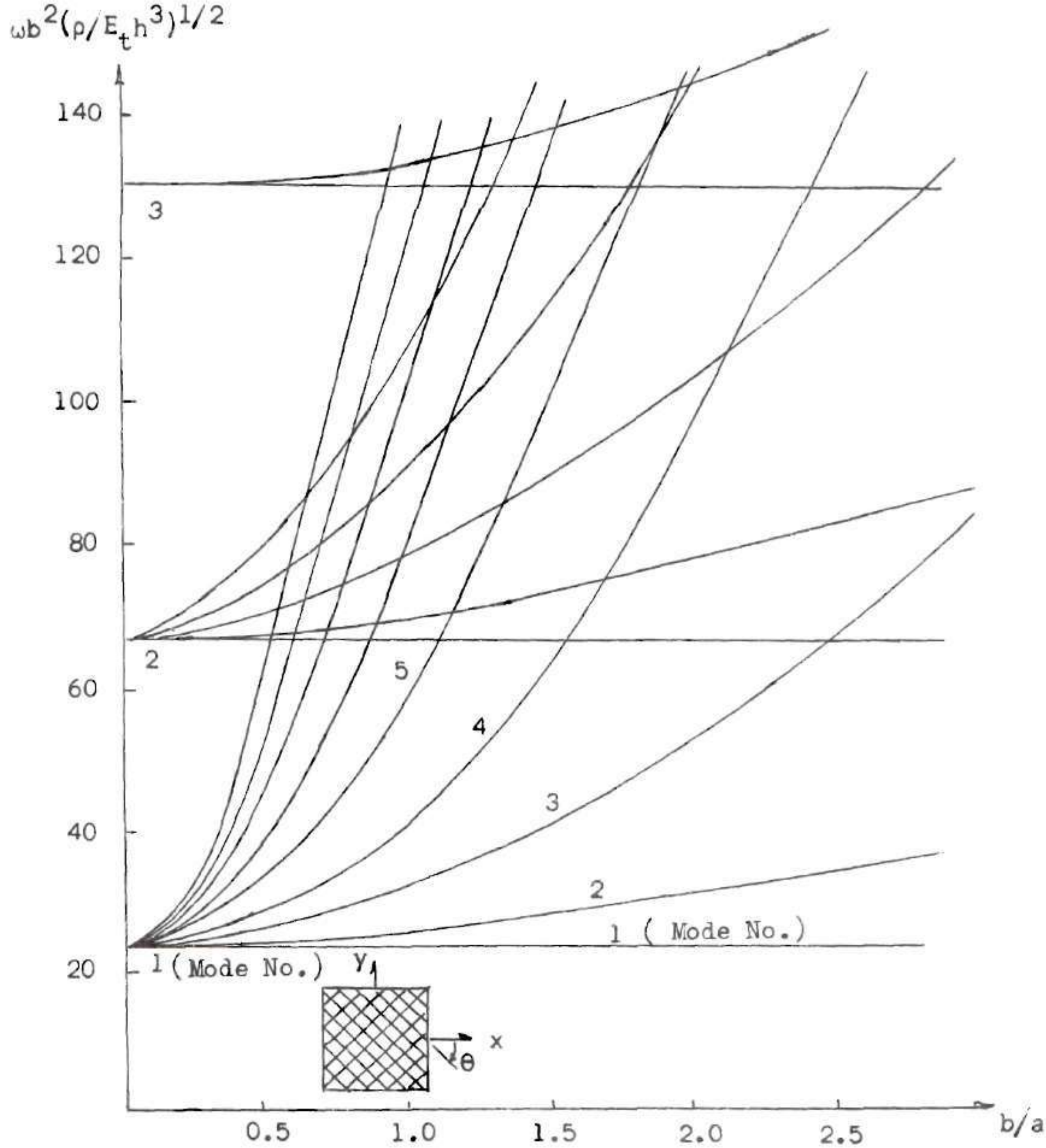

Figure 11. Frequencies of a Two-Layer, 75°/-75°, Angle-Ply Plate with x = 0, a Free and y = 0, b Rigidly-Clamped.

$(E_\ell/E_t = 40, \quad G_{\ell t}/E_t = 0.5, \quad \nu_{\ell t} = 0.25)$

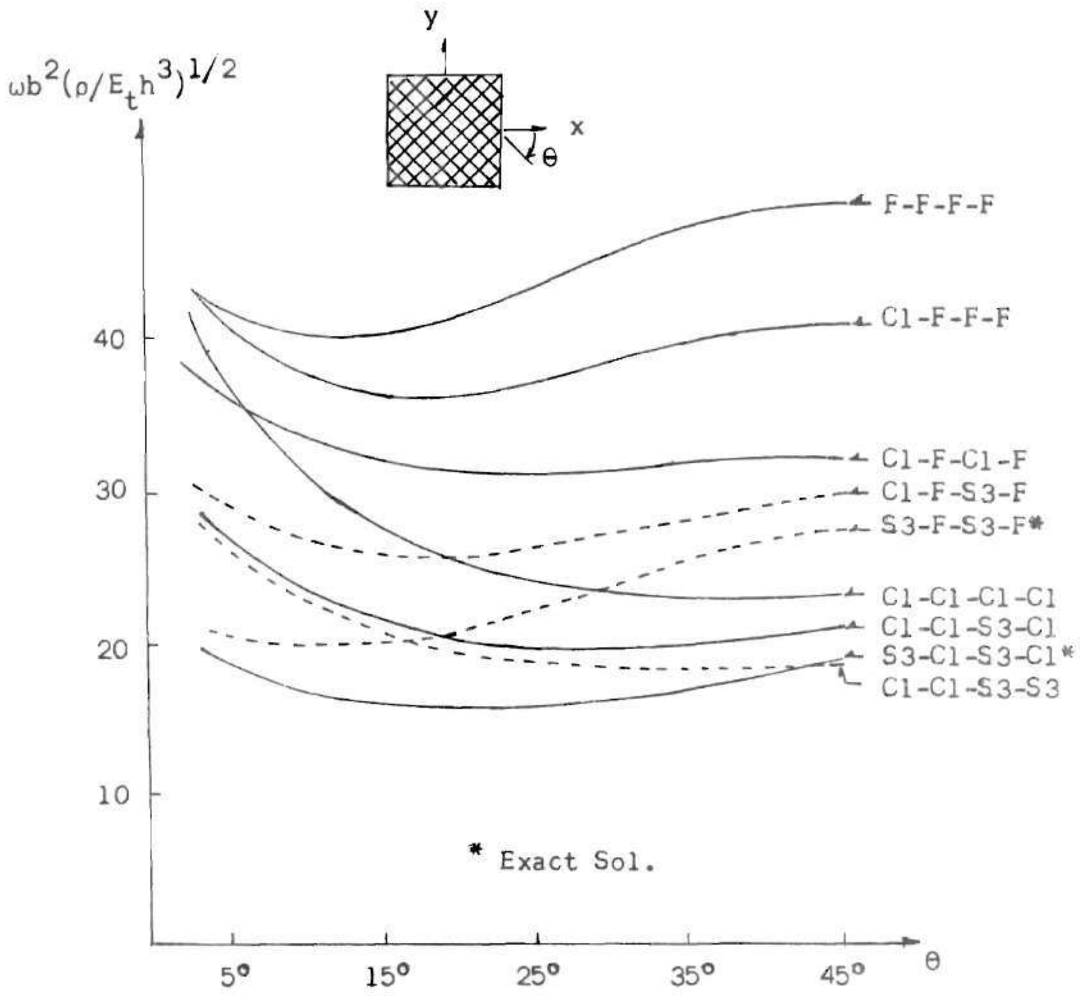

Figure 12. **Frequencies of Mode** $s_x = s_y = 2$ for Two-Layer Angle-Ply Plates with Various Boundary Conditions.

$(E_\ell/E_t = 40, \quad G_{\ell t}/E_t = 0.5, \quad \nu_{\ell t} = 0.25)$

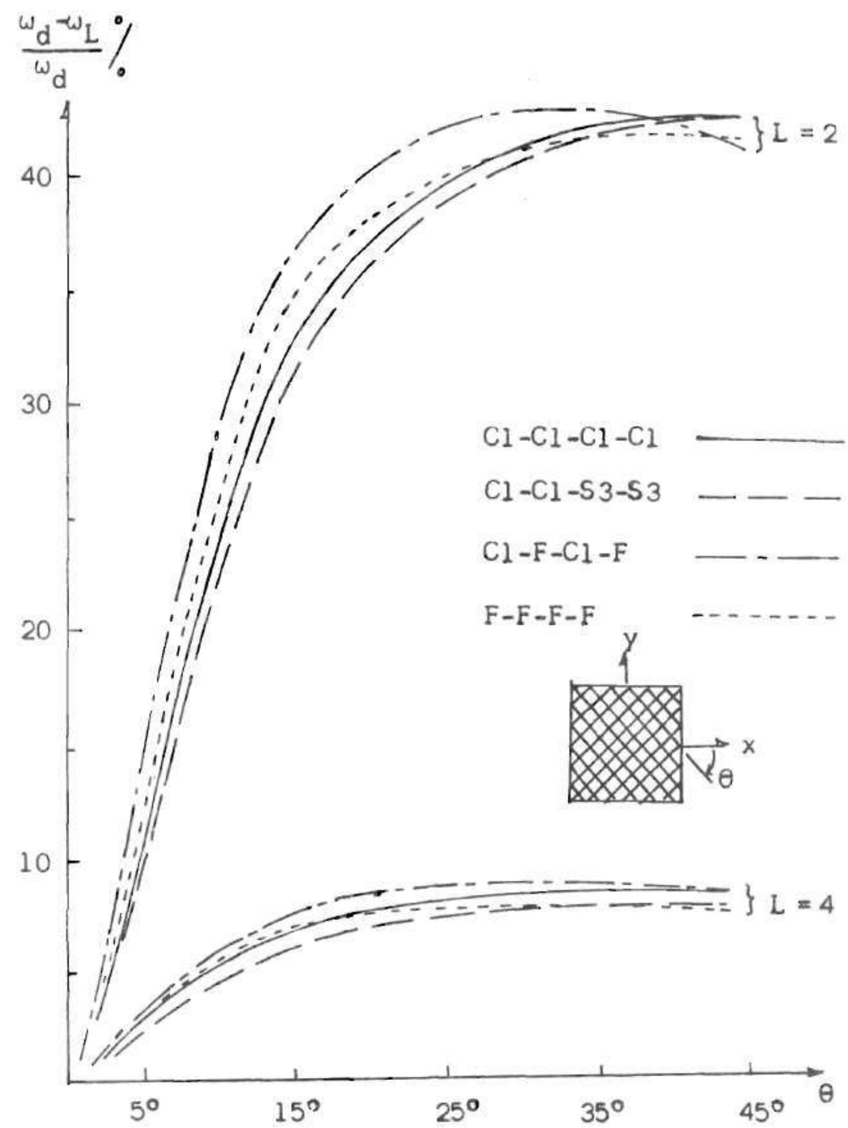

Figure 13. Coupling Effects on Fundamental Frequencies of Angle-Ply Square Plates with Various Boundary Conditions.

(E_ℓ/E_t = 40, $G_{\ell t}/E_t$ = 0.5, $\nu_{\ell t}$ = 0.25)

L indicates the total number of layers in the plate.

APPENDIX A

STIFFNESS CONSTANTS OF A LAMINA REFERRED TO GEOMETRIC AXES

An orthotropic composite lamina, in the two-dimensional case, can be described by the engineering constants E_ℓ and E_t — elastic moduli in the longitudinal and lateral directions, respectively, $G_{\ell t}$ — the shearing modulus, $\nu_{\ell t}$ — Poisson's ratio giving the lateral strain caused by a strain in the longitudinal direction, and $\nu_{t\ell}$ — Poisson's ratio giving the longitudinal strain caused by a strain in the lateral direction. These orthotropic engineering constants can be predicted in terms of the constituent material properties and the phase geometry by micromechanical techniques accomplished by several investigators, e.g. [31, 32, 33]. If the lamina with principal material directions designated by the longitudinal direction ℓ (parallel to the filaments) and the lateral direction t, the stress-strain relations referred to principal material axes are

$$\begin{bmatrix} \sigma_\ell \\ \sigma_t \\ \tau_{\ell t} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_\ell \\ \varepsilon_t \\ \gamma_{\ell t} \end{bmatrix} \tag{A-1}$$

The four constants C_{ij} are given by

$$C_{11} = \frac{E_\ell}{1 - \nu_{\ell t}\nu_{t\ell}} \qquad \text{(A-2)}$$

$$C_{12} = \frac{E_t \nu_{t\ell}}{1 - \nu_{\ell t}\nu_{t\ell}} = \frac{E_\ell \nu_{\ell t}}{1 - \nu_{\ell t}\nu_{t\ell}}$$

$$C_{22} = \frac{E_t}{1 - \nu_{\ell t}\nu_{t\ell}}$$

$$C_{66} = G_{\ell t}$$

For an orthotropic lamina loaded in its plane if the principal material direction ℓ (direction of fibers) makes an angle θ (positive for clockwise rotation) with the geometric axis x, the stiffness constants C^*_{ij} referred to geometric axes can be obtained by the transformation of stresses and strains [18], and thus are given by

$$C^*_{11} = C_{11} \cos\theta + 2(C_{12} + 2C_{66}) \sin^2\theta \cos^2\theta + C_{22} \sin^4\theta \qquad \text{(A-3)}$$

$$C^*_{12} = (C_{11} + C_{22} - 4C_{66}) \sin^2\theta \cos^2\theta + C_{12}(\sin^4\theta + \cos^4\theta)$$

$$C^*_{16} = (C_{11} - C_{12} - 2C_{66}) \sin\theta \cos^3\theta + (C_{12} - C_{22} + 2C_{66}) \sin^3\theta \cos\theta$$

$$C^*_{22} = C_{11} \sin^4\theta + 2(C_{12} + 2C_{66}) \sin^2\theta \cos^2\theta + C_{22} \cos^4\theta$$

$$C^*_{26} = (C_{11} - C_{12} - 2C_{66}) \sin^3\theta \cos\theta + (C_{12} - C_{22} + 2C_{66}) \sin\theta \cos^3\theta$$

$$C^*_{66} = (C_{11} + C_{22} - 2C_{12} - 2C_{66}) \sin^2\theta \cos^2\theta + C_{66}(\sin^4\theta + \cos^4\theta)$$

APPENDIX B

DERIVATION OF EQUATIONS OF MOTION

Hamilton's principle states that the time integral of the Lagrangian function over a time interval t_1 to t_2 is an extremum for the "actual" motion with respect to all admissible virtual displacements which vanish at instants of time t_1 and t_2 at all points of the body and over the boundaries where the displacements are prescribed throughout the entire time interval. i.e.

$$\delta \int_{t_1}^{t_2} L \, dt = 0 \tag{B-1}$$

where

$$L = U_s - T$$

Now

$$\delta U_s = \int_0^b \int_0^a [N_{xx} \delta u^o_{,x} + N_{yy} \delta v^o_{,y} + N_{xy} (\delta u^o_{,y} + \delta v^o_{,x}) - M_{xx} \delta W_{,xx} - M_{yy} \delta W_{,yy} - 2M_{xy} \delta W_{,xy}] \, dxdy \tag{B-2}$$

$$\delta T = \int_0^b \int_0^a [\rho (\dot{u}^o \delta \dot{u}^o + \dot{v}^o \delta \dot{v}^o + \dot{w} \delta \dot{w}) - Q(\dot{u}^o \delta \dot{w}_{,x} + \dot{w}_{,x} \delta \dot{u}^o + \dot{v}^o \delta \dot{w}_{,y} + \dot{w}_{,y} \delta \dot{v}^o) + I(\dot{w}_{,x} \delta \dot{w}_{,x} + \dot{w}_{,y} \delta \dot{w}_{,y})] \, dxdy$$

Performing integration by parts, we may write

$$\delta U_s = -\int_0^b\int_0^a (N_{xx,x}\delta u^o + N_{yy,y}\delta v^o + N_{xy,y}\delta u^o + N_{xy,x}\delta v^o \tag{B-3}$$

$$+ M_{xx,xx}\delta w + M_{yy,yy}\delta w + 2M_{xy,xy}\delta w)\ dxdy$$

$$+ \int_0^b (N_{xx}\delta u^o)\Big|_0^a dy + \int_0^a (N_{yy}\delta v^o)\Big|_0^b dx + \int_0^a (N_{xy}\delta u^o)\Big|_0^b dx$$

$$+ \int_0^b (N_{xy}\delta v^o)\Big|_0^a dy - \int_0^b (M_{xx}\delta w,_x)\Big|_0^a dy + \int_0^b (M_{xx,x}\delta w)\Big|_0^a dy$$

$$- \int_0^a (M_{yy}\delta w,_y)\Big|_0^b dx + \int_0^a (M_{yy,y}\delta w)\Big|_0^b dx - 2(M_{xy}\delta w)\Big|_0^a\Big|_0^b$$

$$+ \int_0^b (2M_{xy,y}\delta w)\Big|_0^a dy + \int_0^a (2M_{xy,x}\delta w)\Big|_0^b\ dx$$

$$\int_{t_1}^{t_2}\delta T dt = -\int_{t_1}^{t_2}\int_0^b\int_0^a [\rho(\ddot{u}^o\delta u^o + \ddot{v}^o\delta v^o + \ddot{w}\delta w) - Q(\ddot{w},_x\delta u^o \tag{B-4}$$

$$- \ddot{w},_y\delta v^o - \ddot{u}^o,_x\delta w - \ddot{v}^o,_y\delta w) + I(\ddot{w},_{xx} + \ddot{w},_{yy})\delta w]\ dxdydt$$

$$+ \int_0^b\int_0^a [\rho(\dot{u}^o\delta u^o + \dot{v}^o\delta v^o + \dot{w}\ \delta w) - Q(\dot{u}^o\delta w,_x + \dot{w},_x\delta u^o$$

$$+ \dot{v}^o\delta w,_y + \dot{w},_y\delta v^o) + I(\dot{w},_x\delta w,_x + \dot{w},_y\delta w,_y)]\Big|_{t_1}^{t_2} dxdy$$

$$+ \int_{t_1}^{t_2}\int_0^b Q(\ddot{u}^o\delta w)\Big|_0^a\ dydt + \int_{t_1}^{t_2}\int_0^a Q(\ddot{v}^o\delta w)\Big|_0^b\ dxdt$$

$$- \int_{t_1}^{t_2}\int_0^b I(\ddot{w},_x\delta w)\Big|_0^a\ dydt - \int_{t_1}^{t_2}\int_0^a I(\ddot{w},_y\delta w)\Big|_0^b\ dxdt$$

Substituting Eqs. (B-2) - (B-4) into (B-1) yields

$$0 = -\int_{t_1}^{t_2}\int_0^b\int_0^a \Big\{[N_{xx,x}+N_{xy,y} - \rho\ddot{u}^o + Q\ddot{w},_x]\delta u^o + [N_{xy,x}+N_{yy,y} \tag{B-5}$$

$$- \rho\ddot{v}^o + Q\ddot{w},_y]\,\delta v^o + [M_{xx,xx}+2M_{xy,xy}+M_{yy,yy} - \rho\ddot{w} - Q(\ddot{u}^o,_x+\ddot{v}^o,_y)$$

$$+ I(\ddot{w},_{xx}+\ddot{w},_{yy})]\delta w\Big\}dxdydt + \int_{t_1}^{t_2}\int_0^b [N_{xx}\,\delta u^o + N_{xy}\,\delta v^o$$

$$+ (M_{xx,x}+2M_{xy,y} - Q\ddot{u}^o + I\ddot{w},_x)\delta w - M_{xx}\delta w,_x]\Big|_0^a\, dydt + \int_{t_1}^{t_2}\int_0^a [N_{xy}\delta u^o$$

$$+ N_{yy}\delta v^o + (M_{yy,y} + 2M_{xy,x} - Q\ddot{v}^o + I\ddot{w},_y)\delta w - M_{yy}\delta w,_y]\Big|_0^b\, dxdt$$

$$- \int_{t_1}^{t_2}(2M_{xy}\delta w)\Big|_0^a\Big|_0^b dt - \int_0^b\int_0^a [(\rho\dot{u}^o - Q\dot{w},_x)\delta u^o + (\rho\dot{v}^o - Q\dot{w},_y)\delta v^o$$

$$+ \rho\dot{w}\delta w - (Q\dot{u}^o - I\dot{w},_x)\delta w,_x - (Q\dot{v}^o - I\dot{w},_y)\,\delta w,_y]\Big|_{t_1}^{t_2}\, dxdy$$

Since the variations u^o, v^o and w are arbitrary in the region $0 < x < a$, $0 < y < b$; their coefficients in the first integral in (B-5) must vanish which imply the equations of motion

$$N_{xx,x} + N_{xy,y} = \rho\ddot{u}^o - Q\ddot{w},_x \tag{B-6}$$

$$N_{xy,x} + N_{yy,y} = \rho\ddot{v}^o - Q\ddot{w},_y \tag{B-7}$$

$$M_{xx,xx} + 2M_{xy,xy} + M_{yy,yy} = \rho\ddot{w} + Q(\ddot{u}^o,_x + \ddot{v}^o,_y) - I(\ddot{w},_{xx}+\ddot{w},_{yy}) \tag{B-8}$$

The integrands in the other integrals in (B-5) vanish to yield the boundary conditions:

(i) along $x = 0$ and $x = a$

$$\text{either } u^o = 0 \quad \text{or} \quad N_{xx} = 0 \tag{B-9}$$

$$v^o = 0 \qquad N_{xy} = 0$$

$$w = 0 \qquad M_{xx,x} + 2M_{xy,y} - Q\ddot{u}^o + I\ddot{w},_x = 0$$

$$w,_x = 0 \qquad M_{xx} = 0$$

(ii) along $y = 0$ and $y = b$

$$\text{either } v^o = 0 \quad \text{or} \quad N_{yy} = 0 \tag{B-10}$$

$$u^o = 0 \qquad N_{xy} = 0$$

$$w = 0 \qquad M_{yy,y} + 2M_{xy,x} - Q\ddot{v}^o + I\ddot{w},_y = 0$$

$$w,_y = 0 \qquad M_{yy} = 0$$

(iii) at corners: either $w = 0$ or $M_{xy} = 0$

and the initial conditions:

$$\text{either} \quad u^o = v^o = w = w,_x = w,_y = 0 \tag{B-12}$$

$$\text{or} \quad \rho\dot{u}^o - Q\dot{w},_x = \rho\dot{v}^o - Q\dot{w},_y = \dot{w} = Q\dot{u}^o - I\dot{w},_x = Q\dot{v}^o - I\dot{w},_y = 0$$

APPENDIX C

PROPERTIES OF THE DYNAMIC EDGE EFFECTS FOR LAMINATED PLATES

For antisymmetric cross-ply laminates it can be shown that

$$A_{22} = A_{11},\ B_{22} = -B_{11},\ D_{22} = D_{11},\ B_{12} = B_{66} = 0 \qquad \text{(C-1)}$$

Then coefficients in the characteristic equation $\Delta_1(\gamma^2) = 0$ become

$$-P_o = A_{66}(A_{11}D_{11} - B_{11}^2) \qquad \text{(C-2)}$$

$$P_1 = k_2^2[D_{11}(A_{11}^2 - A_{12}^2 - 2A_{12}A_{66}) + 2A_{11}A_{66}(D_{12} + 2D_{66}) - A_{11}B_{11}^2]$$

$$-P_3 = k_2^6[(\lambda/k_2^4 - D_{11})(A_{11}^2 - A_{12}^2 - 2A_{12}A_{66}) - 2A_{11}A_{66}(D_{12} + 2D_{66}) + A_{11}B_{11}^2]$$

$$P_4 = k_2^8 A_{66}[(\lambda/k_2^4 - D_{11})A_{11} + B_{11}^2]$$

and

$$\lambda/k_2^4 - D_{11} = D_{11}\mu^4 + 2(D_{12} + 2D_{66})\mu^2 - \frac{B_{11}^2[A_{66}\mu^8 + A_{11}\mu^6 + 2(A_{12} + A_{66})\mu^4 + A_{11}\mu^2 + A_{66}]}{A_{11}A_{66}\mu^4 + (A_{11}^2 - A_{12}^2 - 2A_{12}A_{66})\mu^2 + A_{11}A_{66}}$$

where $\mu = \frac{k_1}{k_2}$ $(k_1,\ k_2 \neq 0)$. Coefficients in the equation $\Delta_1^*(-\xi) = 0$ are given by

$$P_o^* = -P_o \qquad \text{(C-3)}$$

$$P_1^* = P_1 + P_o^* k_1^2$$

$$P_2^* = (P_3^* - P_3)/k_1^2$$

$$P_4^* = P_4/k_1^2$$

By Schwartz inequality

$$\int_{x_1}^{x_2} f(x)g(x)dx \leq \left\{\int_{x_1}^{x_2} f^2 dx\right\}^{1/2} \left\{\int_{x_1}^{x_2} g^2 dx\right\}^{1/2}$$

it can be proved that

$$A_{11}D_{11} - B_{11}^2 > 0 \;, \qquad \boxed{P_o^* > 0}$$

By algebraic performance, we can obtain

$$\begin{aligned} P_3^* = \frac{k_2^6 A_{66}}{J_1^*} \Big\{ & A_{11}A_{66}(A_{11}D_{11} - B_{11}^2)\mu^6 + A_{11}[D_{11}(A_{11}^2 - A_{12}^2 - 2A_{12}A_{66}) \\ & + 2A_{11}A_{66}(D_{12} + 2D_{66}) - A_{11}B_{11}^2]\mu^4 + A_{11}[A_{66}(A_{11}D_{11} - B_{11}^2) \\ & + 2(D_{12} + 2D_{66})(A_{11}^2 - A_{12}^2 - 2A_{12}A_{66}) - 2A_{12}B_{11}^2]\mu^2 \\ & + [2A_{11}^2A_{66}(D_{12} + 2D_{66}) - B_{11}^2A_{12}(A_{12} + 2A_{66})] \Big\} \end{aligned} \quad \text{(C-4)}$$

where $J_1^* = A_{11}A_{66}\mu^4 + (A_{11}^2 - A_{12}^2 - 2A_{12}A_{66})\mu^2 + A_{11}A_{66} > 0$

and

$$\begin{aligned} P_2^* = \frac{k_2^4}{J_1^*} \Big\{ & A_{66}(A_{11}D_{11} - B_{11}^2)(A_{11}^2 - A_{12}^2 - 2A_{12}A_{66})\mu^6 + [(A_{11}^2 - A_{12}^2 \\ & - 2A_{12}A_{66})(D_{11}(A_{11}^2 - A_{12}^2 - 2A_{12}A_{66}) + 2A_{11}A_{66}(D_{12} + 2D_{66}) \\ & - A_{11}B_{11}^2) + A_{11}^2A_{66}(A_{11}D_{11} - B_{11}^2)]\mu^4 + (A_{11}^2 - A_{12}^2 - 2A_{12}A_{66})[A_{66}(A_{11}D_{11} \\ & - B_{11}^2) + 2(D_{12} + 2D_{66})(A_{11}^2 - A_{12}^2 - 2A_{12}A_{66}) - 2A_{12}B_{11}^2]\mu^2 \end{aligned} \quad \text{(C-5)}$$

$$+ A_{11}A_{66}[A_{66}(A_{11}D_{11} - B_{11}^2) + 2(D_{12} + 2D_{66})(A_{11}^2 - A_{12}^2$$

$$- 2A_{12}A_{66}) - 2A_{12}B_{11}^2]\}$$

Since

$$A_{11}^2 - A_{12}^2 - 2A_{12}A_{66} = \left[\int_o^{h/2} (C_{11} + C_{22})dz\right]^2 - 4\int_o^{h/2} C_{12}dz \int_o^{h/2} (C_{12}$$

$$+ 2C_{66})dz > 0$$

hence for an antisymmetric cross-ply plate satisfying the following two conditions

$$A_{66}(A_{11}D_{11} - B_{11}^2) + 2(D_{12} + D_{66})(A_{11}^2 - A_{12}^2 - 2A_{12}A_{66}) - 2A_{12}B_{11}^2 > 0 \quad \text{(C-6)}$$

$$2A_{11}^2 A_{66}(D_{12} + 2D_{66}) - B_{11}^2 A_{12}(A_{12} + 2A_{66}) > 0 \quad \text{(C-7)}$$

all coefficients P_s^* ($s = 0, 1, 2, 3$) will be definitely positive and the dynamic edge effect will never degenerate. Conditions (C-6) and (C-7) are always satisfied for a cross-ply plate consisting of similar orthotropic layers. If the material properties of individual layers in the cross-ply plate are the same, then

$$A_{11} = \frac{h}{2}(C_{11} + C_{22}), \quad A_{12} = C_{12}h, \quad A_{66} = C_{66}h \quad \text{(C-8)}$$

$$B_{11}^2 \le \frac{h^4}{64}(C_{11} - C_{22})^2$$

$$D_{ij} = \frac{h^2}{12} A_{ij}$$

Thus

$$A_{66}(A_{11}D_{11} - B_{11}^2) + 2(D_{12} + 2D_{66})(A_{11}^2 - A_{12}^2 - 2A_{12}A_{66}) - 2A_{12}B_{11}^2$$

$$\geq \frac{h^5}{192}(2C_{12} + C_{66})(C_{11}^2 + 14C_{11}C_{22} + C_{22}^2) - \frac{h^5}{6}C_{12}^2(C_{12} + 2C_{66})$$

$$+ \frac{h^5}{12}C_{66}(C_{11} + C_{22})^2 - \frac{h^5}{3}C_{12}C_{66}(C_{12} + 2C_{66}) > 0$$

since $C_{11} > C_{22} > 2C_{12}$ and $C_{11} + C_{22} > C_{12} + 2C_{66}$. Also

$$2A_{11}^2A_{66}(D_{12} + 2D_{66}) - B_{11}^2A_{12}(A_{12} + 2A_{66})$$

$$\geq \frac{h^6}{192}(C_{12} + 2C_{66})[8C_{66}(C_{11} + C_{22})^2 - 3C_{12}(C_{11} - C_{22})^2] > 0$$

since $8C_{66} > 3C_{12}$ is usually true for composite materials (e.g. boron-epoxy, glass-epoxy, graphite-epoxy, etc.). Using the relation (C-7), we can find that

$$D_{11}(A_{11}^2 - A_{12}^2 - 2A_{12}A_{66}) + 2A_{11}A_{66}(D_{12} + 2D_{66}) - A_{11}B_{11}^2$$

$$> A_{11}D_{11}\left(A_{11} - \frac{A_{12}^2 + 2A_{12}A_{66}}{A_{11}}\right) - B_{11}^2\left(A_{11} - \frac{A_{12}^2 + 2A_{12}A_{66}}{A_{11}}\right)$$

$$= (A_{11}D_{11} - B_{11}^2)\left(A_{11} - \frac{A_{12}^2 + 2A_{12}A_{66}}{A_{11}}\right) > 0$$

Hence

$$\boxed{P_1^* > 0}\ , \quad \boxed{P_2^* > 0}\ , \quad \boxed{P_3^* > 0}$$

Since all coefficients P_s^* in the equation $\Delta_1^*(-\xi) = 0$ are positive, therefore, the equation $\Delta_1^*(-\xi) = 0$ has no positive real or zero roots.

For an antisymmetric angle-ply plate, P_s^* $(s = 0,\ldots,3)$ are given by

$$P_o^* = A_{11}(A_{66}D_{11} - B_{16}^2) \tag{C-9}$$

$$P_1^* = P_o^* k_1^2 + [D_{11}(A_{11}A_{22} - A_{12}^2 - 2A_{12}A_{66}) + 2A_{11}A_{66}(D_{12} + 2D_{66})$$
$$- 6A_{11}B_{16}B_{26} + 2(3A_{12} - 2A_{66})B_{16}^2]$$

$$P_2^* = \frac{P_3^*}{k_1^2} - k_2^4[(\frac{\lambda}{k_2^4} - D_{22})(A_{11}A_{22} - A_{12}^2 - 2A_{12}A_{66}) - 2A_{22}A_{66}(D_{12} + D_{66})$$
$$+ 6A_{22}B_{16}B_{26} - 2(3A_{12} - 2A_{66})B_{26}^2]\,\mu^2$$

$$P_3^* = k_2^6 A_{22}[(\frac{\lambda}{k_2^4} - D_{22})A_{66} + B_{26}^2]\,\mu^2$$

$$\frac{\lambda}{k_2^4} - D_{22} = D_{11}\mu^4 + 2(D_{12} + 2D_{66})\mu^2 - \frac{1}{J_1^*}[(A_{11}\mu^2 + A_{66})(B_{16}\mu^2 + 3B_{26})^2\,\mu^2$$
$$-2(A_{12} + A_{66})(B_{16}\mu^2 + 3B_{26})(3B_{16}\mu^2 + B_{26})\mu^2 + (A_{66}\mu^2$$
$$+ A_{22})(3B_{16}\mu^2 + B_{26})^2]$$

Since

$$C_{11}^* C_{66}^* - C_{16}^{*2} = (C_{11}C_{22} - C_{12}^2 - 2C_{12}C_{66})\sin^2\theta\cos^2\theta + C_{11}C_{66}\cos^4\theta$$
$$+ C_{22}C_{66}\sin^4\theta > 0$$

hence

$$A_{66}D_{11} = \int C_{66}^*\,dz \int C_{11}^* z^2 dz > \left[\int \sqrt{C_{11}^* C_{66}^*}\;z dz\right]^2 > \left[\int C_{16}^* z dz\right]^2 = B_{16}^2$$

which implies that $\boxed{P_o^* > 0}$

If the plate consists of even number of identical orthotropic laminas with alternating angles of orientations between adjacent layers, then

$$A_{11} = C^*_{11}h,\ A_{22} = C^*_{22}h,\ A_{12} = C^*_{12}h,\ A_{66} = C^*_{66}h,\ D_{ij} = \frac{h^2}{12} A_{ij}\ ,$$

$$B^2_{16} \leq \frac{h^4}{16} C^{*2}_{16}\ ,\ B^2_{26} \leq \frac{h^4}{16} C^{*2}_{26}\ ,\quad |B_{16}B_{26}| \leq \frac{h^4}{16} |C^*_{16}C^*_{26}|$$

By algebraic performance, it can be found that

$$\boxed{P^*_1 > 0}\ ,\quad \boxed{P^*_3 > 0}$$

and the value of $\Delta^*_1(-\xi)$ is always positive whenever ξ is positive under the conditions $C_{66} > C_{12}$ and $100\ C^2_{66} > C_{12}C_{11}$. These conditions are usually satisfied for composite materials (e.g. boron-epoxy, glass-epoxy, graphite-epoxy, etc.). Hence the equation $\Delta^*_1(-\xi) = 0$ has no positive real or zero roots definitely for these specific plates.

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VITA

Chien-Chang Lin was born on July 6, 1941, in Taiwan, China. He received a Bachelor of Science degree in Applied Mathematics from National Chung-Hsing University in June, 1965. During the period from July, 1965, to July 1966, he participated in the Reserve Officer Training Program of the Army of the Republic of China. From 1966 to 1968, he was a mathematical lecturer at Taiwan Provincial Taichung Junior Teacher College.

In September 1968, he entered the Georgia Institute of Technology as a graduate student and also as a research assistant in the School of Engineering Science and Mechanics. In December, 1969, he received the degree of Master of Science in Engineering Mechanics and continued to study for the doctoral degree.

On July 17, 1971, he married Yuan-Kuei Wu of Miaoli, Taiwan.